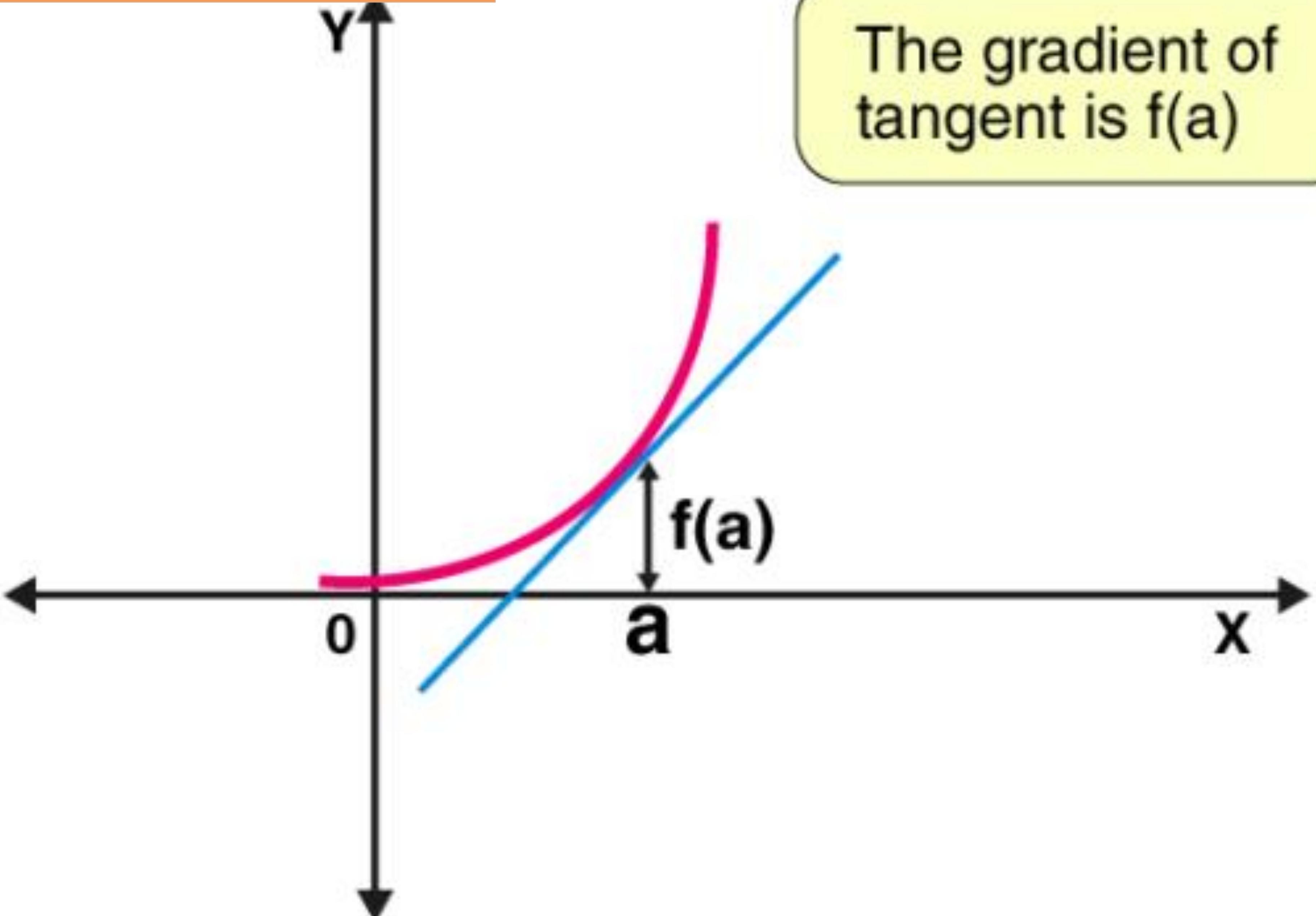
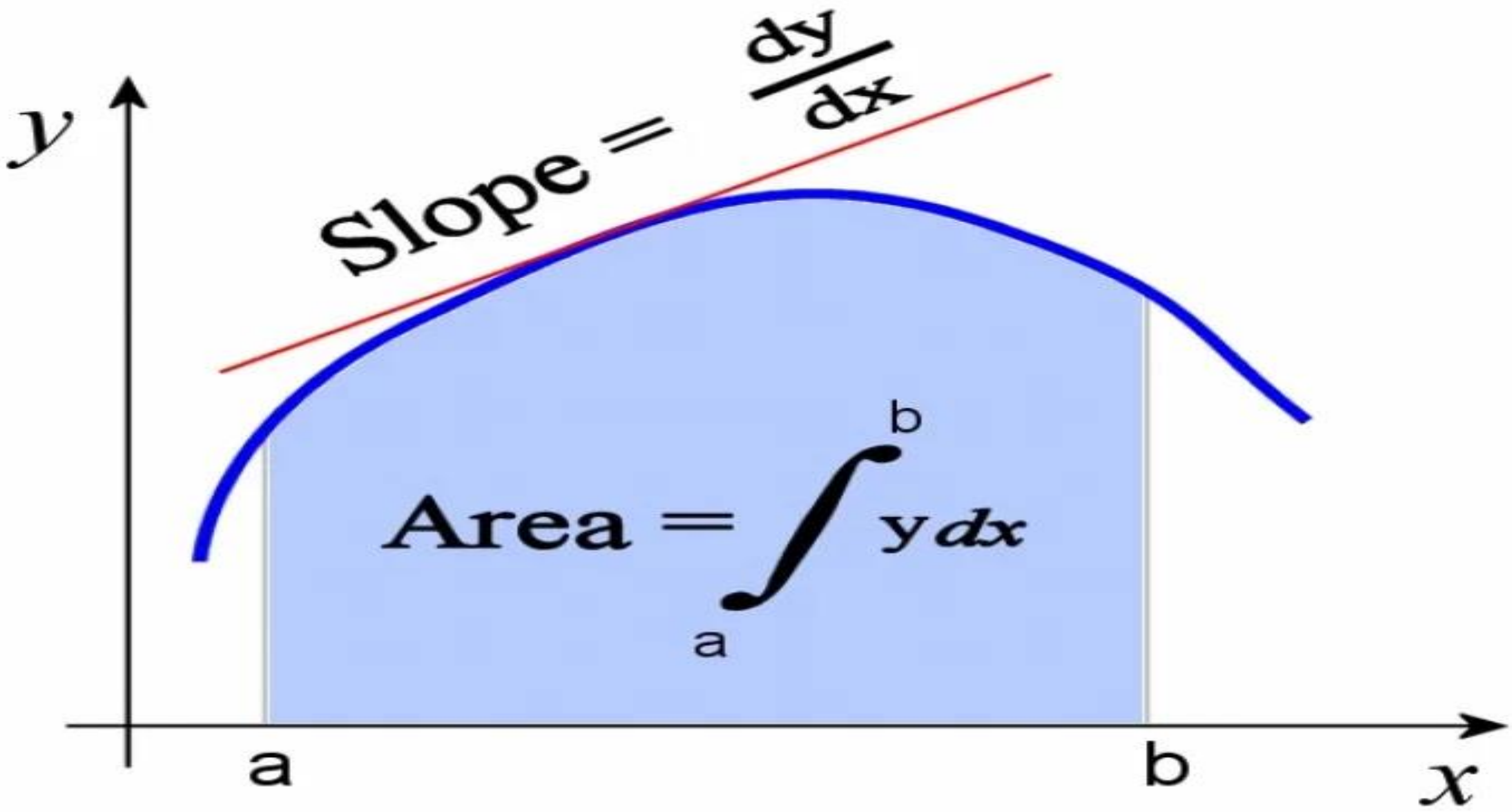
DIFFERENTIATION



The gradient of tangent is f(a)



y
Slope = dy/dx
b
Area = ∫ y dx
a
a
b
x

BASIC DIFFERENTIATION RULES

Rule	Example
CONSTANT RULE: $\frac{d}{dx}(c) = 0$	$y = -5$ $\frac{dy}{dx} =$
POWER RULE: $\frac{d}{dx}(x^n) = nx^{n-1}$	$y = x^5$ $\frac{dy}{dx} =$
CONSTANT MULTIPLE RULE: $\frac{d}{dx}[cf(x)] = cf'(x)$	$y = -5x^2$ $\frac{dy}{dx} =$
SUM RULE: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$	$y = -5x^2 + 4x^3$ $\frac{dy}{dx} =$
DIFFERENCE RULE: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$	$y = 5x^2 - 2x^5$ $\frac{dy}{dx} =$

Differentiation of Trigonometry Functions

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d(\cos x)}{dx} = -\sin x$$

$$\frac{d(\tan x)}{dx} = \sec^2 x$$

$$\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$$

$$\frac{d(\sec x)}{dx} = \sec x \tan x$$

$$\frac{d(\operatorname{cosec} x)}{dx} = -\operatorname{cosec} x \cot x$$

Differentiation of Log and Exponential Function

$$\frac{d(e^x)}{dx} = e^x$$

$$\frac{d(\ln(x))}{dx} = \frac{1}{x}$$

$$\frac{d(a^x)}{dx} = a^x \log a$$

$$\frac{d(x^x)}{dx} = x^x(1 + \ln x)$$

$$\frac{d(\log_a x)}{dx} = \frac{1}{x} \times \frac{1}{\ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left[\cos^{-1}\left(\frac{x}{a}\right)\right] = -\frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{a}{a^2 + x^2}$$

$$\frac{d}{dx}\left[\csc^{-1}\left(\frac{x}{a}\right)\right] = -\frac{a}{x\sqrt{x^2 - a^2}}$$

$$\frac{d}{dx}\left[\sec^{-1}\left(\frac{x}{a}\right)\right] = \frac{a}{x\sqrt{x^2 - a^2}}$$

$$\frac{d}{dx}\left[\cot^{-1}\left(\frac{x}{a}\right)\right] = -\frac{a}{a^2 + x^2}$$

Differentiation Rules

1. $\frac{d}{dx}[cu] = cu'$

2. $\frac{d}{dx}[u \pm v] = u' \pm v'$

3. $\frac{d}{dx}[uv] = uv' + vu'$

4. $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' + uv'}{v^2}$

5. $\frac{d}{dx}[v] = 0$

6. $\frac{d}{dx}\left[u^n\right] = nu^{n-1} \cdot u'$

7. $\frac{d}{dx}[x] = 1$

8. $\frac{d}{dx}\left[|u|\right] = \frac{u}{|u|}(u')$

9. $\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot u'$

10. $\frac{d}{dx}\left[e^u\right] = e^u \cdot u'$

11. $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \cdot u'$

12. $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u \cdot u'$

Chain Rule

If f and g are both differentiable and $F(x)$ is the composite function defined by $F(x) = f(g(x))$ then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))\, g'(x)$$

Differentiate outer function

Differentiate inner function

$$f'(x) = (\ln(x^2 - 1))' = \frac{1}{x^2 - 1}(x^2 - 1)' = \frac{1}{x^2 - 1}2x$$

- keep the inside
- take derivative of outside

multiply by derivative of the inside

$$(f(x))^n \qquad n(f(x))^{n-1} \quad f'(x)$$

$$f(g(x)) \qquad f'(g(x))g'(x)$$

Taking Derivatives Using The Chain Rule

$$F(x) = (x^2 + 1)^{1/2}$$

The Chain Rule

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

$$F'(x) = \frac{x}{\sqrt{x^2 + 1}}$$

Because $f(x) = (5x^2 + 3x - 1)^{1/2}$

$$f'(x) = \frac{1}{2}(5x^2 + 3x - 1)^{-1/2}(10x + 3)$$

$$= \frac{10x + 3}{2\sqrt{5x^2 + 3x - 1}}$$

$$f'(2) = \frac{10 \cdot 2 + 3}{2\sqrt{5(2)^2 + 3 \cdot 2 - 1}}$$

$$= \frac{23}{2\sqrt{25}}$$

$$= \frac{23}{10}$$

$f(x)=\sqrt{x+3}$

$f(x)=(x+3)^{1/2}$ Rewrite as an exponent

$f'(x)=\frac{1}{2}(x+3)^{(1/2-1)}\frac{d}{dx}[x+3]$ Apply the chain rule

$f'(x)=\frac{1}{2}(x+3)^{-1/2}(1)$ Differentiate

$f'(x)=\frac{1}{2\sqrt{x+3}}$ Rewrite to remove the negative exponent

$$f(x) = \frac{1}{3x+3}$$

$$= (3x+3)^{-1}$$

$$f'(x) = -3(3x+3)^{-2}$$

$$f'(x) = \frac{-3}{(3x+3)^2}$$

If $f(2)=4,\quad f'(2)=1,$ then $\lim_{x\to 2}\dfrac{xf(2)-2f(x)}{x-2}=$

(a) 1 (b) 2 (c) 3 (d) – 2

Given $f(2)=4, f'(2)=1$

$$\therefore \lim_{x\to 2}\frac{xf(2)-2f(x)}{x-2}=\lim_{x\to 2}\frac{xf(2)-2f(2)+2f(2)-2f(x)}{x-2}=\lim_{x\to 2}\frac{(x-2)f(2)}{x-2}-\lim_{x\to 2}\frac{2f(x)-2f(2)}{x-2}$$

$$= f(2)-2\lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = f(2)-2f'(2)=4-2(1)=4-2=2$$

Trick : Applying L-Hospital rule, we get $\lim_{x\to 2}\dfrac{f(2)-2f'(2)}{1}=2$.

If $f(a)=3, f'(a)=-2, g(a)=-1, g'(a)=4,$ then $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}=$

(a) -5 (b) 10 (c) -10 (d) 5

$\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$. We add and subtract $g(a)f(a)$ in numerator

$$=\lim_{x\to a}\frac{g(x)f(a)-g(a)f(a)+g(a)f(a)-g(a)f(x)}{x-a}=\lim_{x\to a}f(a)\left[\frac{g(x)-g(a)}{x-a}\right]-\lim_{x\to a}g(a)\left[\frac{f(x)-f(a)}{x-a}\right]$$

$$=f(a)\lim_{x\to a}\left[\frac{g(x)-g(a)}{x-a}\right]-g(a)\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right]=f(a)g'(a)-g(a)f'(a)$$ [by using first principle formula]

$= 3.4 - (-1)(-2) = 12 - 2 = 10$

Trick : $\lim_{x\to a}\frac{g(x)f(a)-g(a)f(x)}{x-a}$

Using L–Hospital's rule, Limit = $\lim_{x\to a}\frac{g'(x)f(a)-g(a)f'(x)}{1}$;

If $5f(x)+3f\left(\frac{1}{x}\right)=x+2$ and $y=xf(x)$ then $\left(\frac{dy}{dx}\right)_{x=1}$ is equal to

(a) 14 (b) (c) 1 (d) None of these

$\because\ 5f(x)+3f\left(\frac{1}{x}\right)=x+2$ ......(i)

Replacing x by $\frac{1}{x}$ in (i), $5f\left(\frac{1}{x}\right)+3f(x)=\frac{1}{x}+2$ ......(ii)

On solving equation (i) and (ii), we get, $16f(x)=5x-\frac{3}{x}+4$, $\therefore\ 16f'(x)=5+\frac{3}{x^2}$

$\because\ y=xf(x)\Rightarrow\frac{dy}{dx}=f(x)+xf'(x)=\frac{1}{16}(5x-\frac{3}{x}+4)+x.\frac{1}{16}(5+\frac{3}{x^2})$

at $x=1$, $\frac{dy}{dx}=\frac{1}{16}(5-3+4)+\frac{1}{16}(5+3)=\frac{7}{8}$.

(i) $\frac{d}{dx}\sin h\,x = \cos h\,x$

(ii) $\frac{d}{dx}\cos h\,x = \sin h\,x$

(iii) $\frac{d}{dx}\tan h\,x = \sec h^2 x$

(iv) $\frac{d}{dx}\cot h\,x = -\operatorname{cosec} h^2 x$

(v) $\frac{d}{dx}\sec h\,x = -\sec h\,x \tan h\,x$

(vi) $\frac{d}{dx}\operatorname{cosec} h\,x = -\operatorname{cosec} h\,x \cot h\,x$

(vii) $\frac{d}{dx}\sin h^{-1}x = 1/\sqrt{(1+x^2)}$

(viii) $\frac{d}{dx}\cos h^{-1}x = 1/\sqrt{(x^2-1)}$

(ix) $\frac{d}{dx}\tan h^{-1}x = 1/(x^2-1)$

(x) $\frac{d}{dx}\cot h^{-1}x = 1/(1-x^2)$

(xi) $\frac{d}{dx}\sec h^{-1}x = -1/x\sqrt{(1-x^2)}$

(xii) $\frac{d}{dx}\operatorname{cosec} h^{-1}x = -1/x\sqrt{(1+x^2)}$

The derivative of $f(x) = |x|^3$ at $x = 0$ is [Rajasthan PET 2001; Haryan

(a) 0 (b) 1 (c) –1 (d) Not defined

$$f(x) = \begin{cases} x^3 & , \quad x \geq 0 \\ -x^3 & , \quad x < 0 \end{cases} \quad \text{and} \quad f'(x) = \begin{cases} 3x^2 & , \quad x \geq 0 \\ -3x^2 & , \quad x < 0 \end{cases}$$

$$f'(0^+) = f'(0^-) = 0$$

The first derivative of the function $(\sin 2x \cos 2x \cos 3x + \log_2 2^{x+3})$ with respect to x at $x = \pi$ is

(a) 2 (b) –1 (c) $-2 + 2^{\pi} \log_e 2$ (d) $-2 + \log_e 2$

$$f(x) = \sin 2x . \cos 2x . \cos 3x + \log_2 2^{x+3}, \quad f(x) = \frac{1}{2} \sin 4x \cos 3x + (x+3) \log_2 2, \quad f(x) = \frac{1}{4}[\sin 7x + \sin x] + x + 3$$

Differentiate *w.r.t. x*,

$$f'(x) = \frac{1}{4}[7\cos 7x + \cos x] + 1, \quad f'(x) = \frac{1}{4} 7\cos 7x + \frac{1}{4}\cos x + 1, \quad f'(\pi) = -2 + 1 = -1.$$

If $y = |\cos x| + |\sin x|$ then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is

(a) $\frac{1-\sqrt{3}}{2}$ (b) 0 (c) $\frac{1}{2}(\sqrt{3}-1)$ (d) None of these

Around $x = \frac{2\pi}{3}$, $|\cos x| = -\cos x$ and $|\sin x| = \sin x$

$$\therefore\ y = -\cos x + \sin x \quad \therefore\ \frac{dy}{dx} = \sin x + \cos x$$

$$\text{At } x = \frac{2\pi}{3},\ \frac{dy}{dx} = \sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{1}{2}(\sqrt{3}-1).$$

$\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$ equals to

(a) 1 (b) $\frac{x^2+1}{x^2-4}$ (c) $\frac{x^2-1}{x^2-4}$ (d) $e^x\frac{x^2-1}{x^2-4}$

Let $y=\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]=\log e^x+\log\left(\frac{x-2}{x+2}\right)^{3/4}$

$$\Rightarrow y=x+\frac{3}{4}[\log(x-2)-\log(x+2)]\Rightarrow\frac{dy}{dx}=1+\frac{3}{4}\left[\frac{1}{x-2}-\frac{1}{x+2}\right]=1+\frac{3}{(x^2-4)}$$

$$\Rightarrow\frac{dy}{dx}=\frac{x^2-1}{x^2-4}.$$

If $f(x)=\log_x(\log x)$, then $f'(x)$ at $x=e$ is

(a)e (b)1/e (c)1 (d)None of these

$$f(x)=\log_x(\log x)=\frac{\log(\log x)}{\log x} \Rightarrow f'(x)=\frac{\frac{1}{x}-\frac{1}{x}\log(\log x)}{(\log x)^2} \Rightarrow f'(e)=\frac{\frac{1}{e}-0}{1}=\frac{1}{e}$$

If $f(x) = |\log x|$, then for $x \neq 1$, $f'(x)$ equals

(a) $\dfrac{1}{x}$ (b) $\dfrac{1}{|x|}$ (c) $\dfrac{-1}{x}$ (d) None of these

$$f(x) = |\log x| = \begin{cases} -\log x, & \text{if } 0 < x < 1 \\ \log x, & \text{if } x \geq 1 \end{cases} \Rightarrow f'(x) = \begin{cases} -\dfrac{1}{x}, & \text{if } 0 < x < 1 \\ \dfrac{1}{x}, & \text{if } x > 1 \end{cases}.$$

Clearly $f'(1^-) = -1$ and $f'(1^+) = 1$, $\therefore$ $f'(x)$ does not exist at $x = 1$

If $x=\exp\left\{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)\right\}$ then $\frac{dy}{dx}$ equals

(a) $2x[1+\tan(\log x)]+x\sec^2(\log x)$

(b) $x[1+\tan(\log x)]+\sec^2(\log x)$

(c) $2x[1+\tan(\log x)]+x^2\sec^2(\log x)$

(d) $2x[1+\tan(\log x)]+\sec^2(\log x)$

$$x=\exp\left\{\tan^{-1}\left(\frac{y-x^2}{x^2}\right)\right\}\Rightarrow \log x=\tan^{-1}\left(\frac{y-x^2}{x^2}\right)$$

$$\Rightarrow \frac{y-x^2}{x^2}=\tan(\log x)\Rightarrow y=x^2\tan(\log x)+x^2\Rightarrow \frac{dy}{dx}=2x.\tan(\log x)+x^2.\frac{\sec^2(\log x)}{x}+2x$$

$$\Rightarrow \frac{dy}{dx}=2x\tan(\log x)+x\sec^2(\log x)+2x\Rightarrow \frac{dy}{dx}=2x[1+\tan(\log x)]+x\sec^2(\log x).$$

If $y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx} =$

(a) 0 (b) $\frac{1}{\sqrt{x}+1}$ (c) 1 (d) None of these

$$y = \sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \cos^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) + \sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \frac{\pi}{2} \Rightarrow \frac{dy}{dx} = 0$$

$$\frac{d}{dx}\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right]$$

(a) $\frac{1}{2(1+x^2)}$ (b) $\frac{1}{1+x^2}$ (c) 1 (d) – 1

$$\frac{d}{dx}\tan^{-1}\left[\frac{\cos x-\sin x}{\cos x+\sin x}\right]=\frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{4}-x\right)\right]=-1\,.$$

If $y = \cos^{-1}\left(\dfrac{5\cos x - 12\sin x}{13}\right)$, $x \in \left(0, \dfrac{\pi}{2}\right)$, then $\dfrac{dy}{dx}$ is equal to

(a) 1 (b) -1 (c) 0 (d) None of these

Let $\cos\alpha = \dfrac{5}{13}$. Then $\sin\alpha = \dfrac{12}{13}$. So, $y = \cos^{-1}\{\cos\alpha.\cos x - \sin\alpha.\sin x\}$

$\therefore y = \cos^{-1}\{\cos(x+\alpha)\} = x + \alpha$ ($\because$ $x + \alpha$ is in the first or the second quadrant)

$$\therefore \frac{dy}{dx} = 1.$$

$\frac{d}{dx}\cosh^{-1}(\sec x) =$

(a) $\sec x$ (b) $\sin x$ (c) $\tan x$ (d) $\operatorname{cosec} x$

We know that $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}}$, $\frac{d}{dx}\cosh^{-1}(\sec x) = \frac{1}{\sqrt{\sec^2 x - 1}}\sec x \tan x = \frac{\sec x \tan x}{\tan x} = \sec x$.

$$\frac{d}{dx}\left[\left(\frac{\tan^2 2x - \tan^2 x}{1-\tan^2 2x\tan^2 x}\right)\cot 3x\right]$$

(a) $\tan 2x \tan x$ (b) $\tan 3x \tan x$ (c) $\sec^2 x$ (d) $\sec x \tan x$

Let $y = \dfrac{\tan^2 2x - \tan^2 x}{1-\tan^2 2x \tan^2 x} = \dfrac{(\tan 2x - \tan x)}{(1+\tan 2x \tan x)}\dfrac{(\tan 2x + \tan x)}{(1-\tan 2x \tan x)} = \tan(2x-x)\tan(2x+x) = \tan x \tan 3x$.

$$\therefore \frac{d}{dx}[y.\cot 3x] = \frac{d}{dx}[\tan x] = \sec^2 x .$$

If $f(x)=\cot^{-1}\left(\frac{x^x-x^{-x}}{2}\right)$, then $f'(1)$ is equal to

(a) -1 (b) 1 (c) $\log 2$ (d) $-\log 2$

$$f(x)=\cot^{-1}\left(\frac{x^x-x^{-x}}{2}\right)$$

Put $x^x=\tan\theta$, $\therefore\ y=f(x)=\cot^{-1}\left(\frac{\tan^2\theta-1}{2\tan\theta}\right)=\cot^{-1}(-\cot 2\theta)=\pi-\cot^{-1}(\cot 2\theta)$

$\Rightarrow y=\pi-2\theta=\pi-2\tan^{-1}(x^x)\Rightarrow \frac{dy}{dx}=\frac{-2}{1+x^{2x}}.x^x(1+\log x)\Rightarrow f'(1)=-1.$

If $y=(1+x)(1+x^2)(1+x^4)\ldots\ldots(1+x^{2^n})$ then $\frac{dy}{dx}$ at $x=0$ is

(a) 1 (b) – 1 (c) 0 (d) None of these

$$y=\frac{(1-x)(1+x)(1+x^2)\ldots\ldots(1+x^{2^n})}{1-x}=\frac{1-x^{2^{n+1}}}{1-x}$$

$$\therefore \frac{dy}{dx}=\frac{-2^{n+1}.x^{2^{n+1}-1}(1-x)+1-x^{2^{n+1}}}{(1-x)^2}, \quad \therefore \text{At } x=0, \ \frac{dy}{dx}=\frac{-2^{n+1}\,0.1+1-0}{1^2}=1.$$

If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, $\frac{dy}{dx} =$

(a) -1 (b) -2 (c) 1 (d) 2

We are given that $xe^{xy} = y + \sin^2 x$

When $x = 0$, we get $y = 0$

Differentiating both sides *w.r.t. x*, we get, $e^{xy} + xe^{xy}\left[x\frac{dy}{dx} + y\right] = \frac{dy}{dx} + 2\sin x\cos x$

Putting, $x = 0$, $y = 0$, we get $\frac{dy}{dx} = 1$.

If $\sin(x+y) = \log(x+y)$, then $\frac{dy}{dx} =$ [Karnataka CET 1993; Rajasthan PET 1989, 1992; Ro

(a) 2 (b) – 2 (c) 1 (d) – 1

$\sin(x+y) = \log(x+y)$

Differentiating with respect to *x*, $\cos(x+y)\left[1+\frac{dy}{dx}\right] = \frac{1}{x+y}\left[1+\frac{dy}{dx}\right]$

$$\left[\cos(x+y) - \frac{1}{x+y}\right]\left[1+\frac{dy}{dx}\right] = 0$$

$\because$ $\cos(x+y) \neq \frac{1}{x+y}$ for any *x* and *y*. So, $1+\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -1$.

Trick: It is an implicit function, so $\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y} = -\frac{\cos(x+y) - \frac{1}{x+y}}{\cos(x+y) - \frac{1}{x+y}} = -1$.

If $\ln(x+y) = 2xy$, then $y'(0) =$

(a) 1 (b) – 1 (c) 2 (d) 0

$$\ln(x+y) = 2xy \Rightarrow \frac{(1+dy/dx)}{(x+y)} = 2\left(x\frac{dy}{dx}+y\right) \Rightarrow \frac{dy}{dx} = \frac{1-2xy-2y^2}{2x^2+2xy-1} \Rightarrow y'(0) = \frac{1-2}{-1} = 1, \text{ at } x=0, \ y=1.$$

If $y = (\sin x)^{\tan x}$, then $\frac{dy}{dx}$ is equal to [IIT 1994; Rajasthan

(a) $(\sin x)^{\tan x}.(1 + \sec^2 x . \log \sin x)$

(b) $\tan x . (\sin x)^{\tan x - 1} . \cos x$

(c) $(\sin x)^{\tan x} ., \sec^2 x \log \sin x$

(d) $\tan x . (\sin x)^{\tan x - 1}$

Given $y = (\sin x)^{\tan x}$

$$\log y = \tan x . \log \sin x$$

Differentiating *w.r.t. x*, $\frac{1}{y} . \frac{dy}{dx} = \tan x . \cot x + \log \sin x . \sec^2 x$

$$\frac{dy}{dx} = (\sin x)^{\tan x}[1 + \log \sin x . \sec^2 x].$$

(i) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+\,.......\infty}}}$, then $y=\sqrt{f(x)+y} \Rightarrow y^2=f(x)+y$

$$2y\frac{dy}{dx}=f'(x)+\frac{dy}{dx}, \quad \therefore \frac{dy}{dx}=\frac{f'(x)}{2y-1}$$

(ii) If $y=f(x)^{f(x)^{f(x)^{f(x)\ldots\ldots\infty}}}$ then $y=f(x)^y$

$$\therefore \log y = y\log f(x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{y.f'(x)}{f(x)}+\log f(x).\frac{dy}{dx}, \quad \therefore \frac{dy}{dx}=\frac{y^2 f'(x)}{f(x)[1-y\log f(x)]}$$

(iii) If $y=f(x)+\cfrac{1}{f(x)+\cfrac{1}{f(x)+\ldots.\infty}}$ then $\dfrac{dy}{dx}=\dfrac{yf'(x)}{2y-f(x)}$

If $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\ldots \text{ to } \infty}}}$ then $\frac{dy}{dx} =$ [Rajasthan PET 2002]

(a) $\frac{x}{2y-1}$ (b) $\frac{2}{2y-1}$ (c) $\frac{-1}{2y-1}$ (d) $\frac{1}{2y-1}$

Solution: (d) $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\ldots \text{ to } \infty}}} \Rightarrow y = \sqrt{x+y} \Rightarrow y^2 = x + y \Rightarrow 2y\frac{dy}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx}(2y-1) = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$

If $y = x^{x^{x \ldots\ldots \infty}}$, then $x(1 - y\log_e x)\frac{dy}{dx}$ is

(a) x^2 (b) y^2 (c) xy^2 (d) None of these

Solution: (b) $y = x^{x^{x \ldots\ldots \infty}} \Rightarrow y = x^y \Rightarrow \log_e y = y\log_e x \Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{y}{x} + \log_e x\,\frac{dy}{dx} \Rightarrow \left(\frac{1}{y} - \log_e x\right)\frac{dy}{dx} = \frac{y}{x} \Rightarrow x(1 - y\log_e x)\frac{dy}{dx} = y^2$

If $y = x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \cfrac{1}{x^2 + \ldots\ldots \infty}}}$, then $\frac{dy}{dx} =$

(a) $\frac{2xy}{2y - x^2}$ (b) $\frac{xy}{y + x^2}$ (c) $\frac{xy}{y - x^2}$ (d) $\frac{2x}{2 + \frac{x^2}{y}}$

Solution: (a) $y = x^2 + \frac{1}{y} \Rightarrow y^2 = x^2 y + 1 \Rightarrow 2y\frac{dy}{dx} = y.2x + x^2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2xy}{2y - x^2}$

Differentiation of Integral Function:-

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)dt = f[g_2(x)]g_2'(x) - f[g_1(x)]g_1'(x) = f[g_2(x)]\frac{d}{dx}g_2(x) - f[g_1(x)]\frac{d}{dx}g_1(x).$$

If $F(x)=\int_{x^2}^{x^3}\log t\,dt\,(x>0)$, then $F'(x)=$

(a) $(9x^2-4x)\log x$ (b) $(4x-9x^2)\log x$ (c) $(9x^2+4x)\log x$ (d) None of these

$$F'(x)=(\log x^3)3x^2-(\log x^2)2x$$

$$=(3\log x)3x^2-2x(2\log x)=9x^2\log x-4x\log x=(9x^2-4x)\log x.$$

If $x = \int_0^y \frac{1}{\sqrt{1+4t^2}} dt$, then $\frac{d^2y}{dx^2}$ is

(a) $2y$ (b) $4y$ (c) $8y$ (d) $6\ y$

$$x = \int_0^y \frac{1}{\sqrt{1+4t^2}} dt \Rightarrow \frac{dx}{dy} = \frac{1}{\sqrt{1+4y^2}} \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{4y}{\sqrt{1+4y^2}} \frac{dy}{dx} \Rightarrow \frac{d^2y}{dx^2} = \frac{4y}{\sqrt{1+4y^2}} \cdot \sqrt{1+4y^2} = 4y$$

3.9 Differentiation of Determinants

Let $\Delta(x) = \begin{vmatrix} a_1(x) & b_1(x) \\ a_2(x) & b_2(x) \end{vmatrix}$. Then $\Delta'(x) = \begin{vmatrix} a_1'(x) & b_1'(x) \\ a_2(x) & b_2(x) \end{vmatrix} + \begin{vmatrix} a_1(x) & b_1(x) \\ a_2'(x) & b_2'(x) \end{vmatrix}$

If we write $\Delta(x) = |\ C_1 C_2 C_3\ |$. Then $\Delta'(x) = |\ C_1'\ C_2\ C_3| + |\ C_1\ C_2'\ C_3| + |\ C_1\ C_2\ C_3'|$

Similarly, if $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} R_1' \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2' \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R_3' \end{vmatrix}$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$ where p is a constant. Then $\frac{d^3}{dx^3}[f(x)]$ at $x=0$ is [IIT 199

(a) p (b) $p+p$ (c) $p+p$ (d) Independent of p

Given

$f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$, 2nd and 3rd rows are constant, so only 1st row will take part in differentiation

$$\therefore \frac{d^3}{dx^3} f(x) = \begin{vmatrix} \frac{d^3}{dx^3} x^3 & \frac{d^3}{dx^3} \sin x & \frac{d^3}{dx^3} \cos x \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$$

We know that $\frac{d^n}{dx^n} x^n = n!, \frac{d^n}{dx^n} \sin x = \sin(x + \frac{n\pi}{2})$ and $\frac{d^n}{dx^n} \cos x = \cos(x + \frac{n\pi}{2})$

Using these results, $$\frac{d^3}{dx^3} f(x) = \begin{vmatrix} 3! & \sin\left(x + \frac{3\pi}{2}\right) & \cos\left(x + \frac{3\pi}{2}\right) \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$$

$$\left.\frac{d^3}{dx^3} f(x)\right|_{\text{at } x=0} = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ 1 & p^2 & p^3 \end{vmatrix}$$ $= 0$ *i.e.*, independent of

If u and v are two functions of x such that their nth derivative exist then

$$D^n(u.v.) = {}^nC_0(D^n u)v + {}^nC_1 D^{n-1}u.Dv + {}^nC_2 D^{n-2}u.D^2v + ... + {}^nC_r D^{n-r}u.D^r v + ... + u.(D^n v).$$

If $y = x^2 e^x$, then value of y_n is

(a) $\{x^2 - 2nx + n(n-1)\}e^x$

(b) $\{x^2 + 2nx + n(n-1)\}e^x$

(c) $\{x^2 + 2nx - n(n-1)\}e^x$

(d) None of these

Applying Leibnitz's theorem by taking x^2 as second function. We get , $D^n y = D^n(e^x.x^2)$

$$= {}^nC_0 D^n(e^x)x^2 + {}^nC_1 D^{n-1}(e^x).D(x^2) + {}^nC_2 D^{n-2}(e^x).D^2(x^2) + ... = e^x.x^2 + ne^x.2x + \frac{n(n-1)}{2!}e^x.2 + 0 + 0 + ...$$

$$y_n = \{x^2 + 2nx + n(n-1)\}e^x.$$

The differential coefficient of $\tan^{-1}\frac{2x}{1-x^2}$ *w.r.t.* $\sin^{-1}\frac{2x}{1+x^2}$ is

[Roorkee 1966; BIT Mesra 1996; Karnataka CET 1994; MP PET 1999; UPSEAT 1999, 2001]

(a) 1 (b) –1 (c) 0 (d) None of these

Let $y_1 = \tan^{-1}\frac{2x}{1-x^2}$ and $y_2 = \sin^{-1}\frac{2x}{1+x^2}$

Putting $x = \tan\theta$

$\therefore\ y_1 = \tan^{-1}\tan 2\theta = 2\theta = 2\tan^{-1}x$ and $y_2 = \sin^{-1}\sin 2\theta = 2\tan^{-1}x$

Again $\frac{dy_1}{dx} = \frac{d}{dx}[2\tan^{-1}x] = \frac{2}{1+x^2}$ ....(i)

and $\frac{dy_2}{dx} = \frac{d}{dx}[2\tan^{-1}x] = \frac{2}{1+x^2}$ ....(ii)

Hence $\frac{dy_1}{dy_2} = 1$

The first derivative of the function $\left[\cos^{-1}\left(\sin\frac{\sqrt{1+x}}{2}\right)+x^x\right]$ with respect to x at $x=1$ is

(a) $\frac{3}{4}$ (b) 0 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

$$f(x)=\cos^{-1}\left[\cos\left(\frac{\pi}{2}-\sqrt{\frac{1+x}{2}}\right)\right]+x^x=\frac{\pi}{2}-\sqrt{\frac{1+x}{2}}+x^x$$

$$\therefore\ f'(x)=-\frac{1}{\sqrt{2}}\cdot\frac{1}{2\sqrt{1+x}}+x^x(1+\log x)\Rightarrow f'(1)=-\frac{1}{4}+1=\frac{3}{4}$$

If , $f(x) = |x-2|$ and $g(x) = f(f(x))$, then for $x>20$, $g'(x)$ equals

(a) -1 (b) 1 (c) 0 (d) None of these

For $x > 20$, we have

$$f(x) = |x-2| = x-2 \quad \text{and, } g(x) = f(f(x)) = f(x-2) = x-2-2 = x-4$$

$$\therefore \; g'(x) = 1$$

If $x = e^{y+e^{y+\ldots\ldots\text{to}\,\infty}}$, then $\dfrac{dy}{dx}$ is

(a) $\dfrac{1+x}{x}$ (b) $\dfrac{1}{x}$ (c) $\dfrac{1-x}{x}$ (d) $\dfrac{x}{1+x}$

$x = e^{y+x}$

Taking log both sides, $\log x = (y+x)\log e = y+x \Rightarrow y+x = \log x \Rightarrow \dfrac{dy}{dx}+1 = \dfrac{1}{x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x} - 1 = \dfrac{1-x}{x}$

(6) **Differentiation by inverse trigonometrical substitution:** For trigonometrical substitutions following formulae and substitution should be remembered

(i) $\sin^{-1} x + \cos^{-1} x = \pi / 2$

(ii) $\tan^{-1} x + \cot^{-1} x = \pi / 2$

(iii) $\sec^{-1} x + cosec^{-1} x = \pi / 2$

(iv) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(v) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{(1-x^2)(1-y^2)}\right]$

(vi) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left[\frac{x \pm y}{1 \mp xy}\right]$

(vii) $2\sin^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})$

(viii) $2\cos^{-1} x = \cos^{-1}(2x^2 - 1)$

(ix) $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(x) $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$

(xi) $3\cos^{-1} x = \cos^{-1}(4x^3 - 3x)$

(xii) $3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$

(xiii) $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$

(xiv) $\sin^{-1}(-x) = -\sin^{-1} x$

(xv) $\cos^{-1}(-x) = \pi - \cos^{-1} x$

(xvi) $\tan^{-1}(-x) = -\tan^{-1} x$ or $\pi - \tan^{-1} x$

(xvii) $\frac{\pi}{4} - \tan^{-1} x = \tan^{-1}\left(\frac{1-x}{1+x}\right)$

Some suitable substitutions

S. N.	Function	Substitution	S. N.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$	(ii)	$\sqrt{x^2 + a^2}$	$x = a\tan\theta$ or $a\cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$	(iv)	$\sqrt{\frac{a-x}{a+x}}$	$x = a\cos 2\theta$
(v)	$\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}$	$x^2 = a^2\cos 2\theta$	(vi)	$\sqrt{ax - x^2}$	$x = a\sin^2\theta$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x = a\tan^2\theta$	(viii)	$\sqrt{\frac{x}{a-x}}$	$x = a\sin^2\theta$
(ix)	$\sqrt{(x-a)(x-b)}$	$x = a\sec^2\theta - b\tan^2\theta$	(x)	$\sqrt{(x-a)(b-x)}$	$x = a\cos^2\theta + b\sin^2\theta$

TRICK NO.1

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$$Y = sinx^{x} + cosx^{tanx}$$

$$y' = sinx^{x}\left[\frac{x}{sinx}cosx + \quad log(sinx)\right] + cosx^{tanx}\left[\frac{tanx}{cosx}(\_sinx) + \quad sec^{2}xlog(cosx)\right]$$

$$y = m^n$$

$$y' = m^n\left[\frac{n}{m} m' + \quad n' \log(m)\right]$$

TRICK NO.2

$$y^2 + x^2 + 2xy = 0$$

$$y' = \frac{-(2x + 2y)}{2y + 2x}$$

TRICK NO.3

$$\frac{d}{dx}\left(\frac{4\log(x)+8}{-7\log(x)+10}\right)$$

$$\frac{96\,\frac{1}{x}}{(10-7\log(x))^2}$$

$$\frac{d}{dx}\left(\frac{af(x)+b}{cf(x)+d}\right)$$

$$\frac{(ad-bc)f'(x)}{(cf(x)+d)^2}$$

$$\frac{d}{dx}\left(\frac{2+30\sin(x)}{\sin(x)-7}\right)$$

$$\frac{-212\cos(x)}{(\sin(x)-7)^2}$$

$$\frac{d}{dx}\left(\frac{3x^2 - 2x + 1}{x^2 + x - 1}\right)$$

$$\frac{5x^2 - 8x + 1}{(x^2 + x - 1)^2}$$

TRICK NO.4

$$\frac{d}{dx}\left(\frac{ax^2+bx+c}{px^2+qx+r}\right)$$

$$\frac{\begin{vmatrix} a & b \\ p & q \end{vmatrix} x^2 + 2\begin{vmatrix} a & c \\ p & r \end{vmatrix} x + \begin{vmatrix} b & c \\ q & r \end{vmatrix}}{(px^2+qx+r)^2}$$

$$\frac{d}{dx}\left(\frac{3sin^2x + 2sinx + 1}{sin^2x + sinx - 1}\right)$$

$$\frac{sin^2x - 8sinx - 3}{(sin^2x + sinx - 1)^2}(cosx)$$

TRICK NO.5

$$\boldsymbol{x^m y^n = (x + y)^{m+n}}$$

$$\frac{dy}{dx} = \frac{d}{dx}(x)\left(\frac{y}{x}\right) = \frac{y}{x}$$

$$x^{16}y^9 = \left(x^2 + y\right)^{17}$$

$$\left(x^2\right)^8 y^9 = \left(x^2 + y\right)^{8+9}$$

$$\frac{dy}{dx} = \frac{y}{x^2}\frac{d}{dx}\left(x^2\right) = \frac{2y}{x}$$

(i) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+\ldots\ldots\infty}}}$, then $y=\sqrt{f(x)+y} \Rightarrow y^2=f(x)+y$

$$2y\frac{dy}{dx}=f'(x)+\frac{dy}{dx}, \quad \therefore \frac{dy}{dx}=\frac{f'(x)}{2y-1}$$

(ii) If $y=f(x)^{f(x)^{f(x)^{f(x)\ldots\infty}}}$ then $y=f(x)^y$

$$\therefore \log y = y\log f(x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{y.f'(x)}{f(x)}+\log f(x).\frac{dy}{dx}, \quad \therefore \frac{dy}{dx}=\frac{y^2 f'(x)}{f(x)[1-y\log f(x)]}$$

(iii) If $y=f(x)+\cfrac{1}{f(x)+\cfrac{1}{f(x)+\ldots\infty}}$ then $\frac{dy}{dx}=\frac{yf'(x)}{2y-f(x)}$