If a function $f(x)$ takes the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = a$, then we say that $f(x)$ is indeterminate or meaningless at $x = a$. Other indeterminate forms are $\infty - \infty, \infty \times \infty, 0 \times \infty, 1^{\infty}, 0^{0}, \infty^{0}$

- In short, we write L.H.L. for left hand limit and R.H.L. for right hand limit.
    - It is not necessary that if the value of a function at some point exists then its limit at that point must exist.

) Method for finding L.H.L. and R.H.L.

(i) For finding right hand limit (R.H.L.) of the function, we write $x + h$ in place of x, while for left hand limit (L.H.L.) we write $x - h$ in place of x.

(ii) Then we replace x by 'a' in the function so obtained.

(iii) Lastly we find limit $h \to 0$.

Existence of limit : $\lim_{x \to a} f(x)$ exists when,

(i) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist *i.e.* L.H.L. and R.H.L. both exists.

(ii) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ *i.e.* L.H.L. = R.H.L.

Sandwich theorem :

If $f(x)$, $g(x)$ and $h(x)$ are any three functions such that,

$f(x) \leq g(x) \leq h(x)$ $\forall x \in$ neighborhood of $x = a$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l\,(\text{say})$,

then $\lim_{x \to a} g(x) = l$. This theorem is normally applied when the $\lim_{x \to a} g(x)$ can't be obtained by using conventional methods as function $f(x)$ and $h(x)$ can be easily found.

If $f(x)=\begin{cases} x, & \text{when } x>1 \\ x^2, & \text{when } x<1 \end{cases}$, then $\lim_{x\to 1} f(x)=$ [MP PET 1987]

(a) x^2 (b) x

(c) -1 (d) 1

To find L.H.L. at $x = 1.$ *i.e.,*

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} (1-h)^2$$

$$= \lim_{h \to 0} (1 + h^2 - 2h) = 1 \quad \text{i.e.,} \quad \lim_{x \to 1^-} f(x) = 1 \text{ ....(i)}$$

Now find R.H.L. at x = 1 i.e., $\lim_{x \to 1^+} f(x) = \lim_{h \to 0} f(1+h) = 1$

i.e., $\lim_{x \to 1^+} f(x) = 1$ .....(ii)

From (i) and (ii), L.H.L. = R.H.L. $\Rightarrow \lim_{x \to 1} f(x) = 1$.

$$\lim_{x \to 2} \frac{|x-2|}{x-2} =$$

(a) 1

(b) –1

(c) Does not exist

(d) None of these

L.H.L.= $\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{h \to 0} \frac{|2-h-2|}{2-h-2} = \lim_{h \to 0} \frac{h}{-h} = -1$ .....(i)

and, R.H.L.= $\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{h \to 0} \frac{|2+h-2|}{2+h-2} = \lim_{h \to 0} \frac{h}{h} = 1$ .....(ii)

From (i) and (ii) L.H.L. $\neq$ R.H.L. *i.e.* $\lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist.

If $f(x)=\begin{cases}\dfrac{2}{5-x}, & \text{when } x<3\\ 5-x, & \text{when } x>3\end{cases}$, then

(a) $\lim_{x\to 3^+} f(x)=0$

(b) $\lim_{x\to 3^-} f(x)=0$

(c) $\lim_{x\to 3^+} f(x)\neq \lim_{x\to 3^-} f(x)$

(d) None of these

$$\lim_{x\to 3+} f(x) = 5 - 3 = 2 \text{ and } \lim_{x\to 3-} f(x) = \frac{2}{5-3} = 1$$

Let the function f be defined by the equation

$$f(x)=\begin{cases}3x, & \text{if } 0\le x\le 1\\ 5-3x, & \text{if } 1<x\le 2\end{cases},\ \text{then}$$ **[SCRA 1996]**

(a) $\lim\limits_{x\to 1} f(x)=f(1)$

(b) $\lim\limits_{x\to 1} f(x)=3$

(c) $\lim\limits_{x\to 1} f(x)=2$

(d) $\lim\limits_{x\to 1} f(x)$ does not exist

$$\text{L.H.L.} = \lim_{x \to 1-0} f(x) = \lim_{h \to 0} f(1-h) = \lim_{h \to 0} 3(1-h) = \lim_{h \to 0} (3-3h) = 3-3.0 = 3$$

$$\text{R.H.L.} = \lim_{x \to 1+0} f(x) = \lim_{h \to 0} f(1+h) = \lim_{h \to 0} [5-3(1+h)] = \lim_{h \to 0} (2-3h) = 2-3.0 = 2$$

Hence $\lim_{x \to 1} f(x)$ does not exists.

$$\lim_{x \to 0} \frac{|x|}{x} =$$ [Roorkee 1982; UPSEAT 2001]

(a) 1

(b) –1

(c) 0

(d) Does not exist

$$\because \lim_{x \to 0-} \frac{|x|}{x} = -1 \text{ and } \lim_{x \to 0+} \frac{|x|}{x} = 1,$$

hence limit does not exists.

Fundamental Theorems on Limits

The following theorems are very useful for evaluation of limits

if $\lim_{x \to 0} f(x) = l$ and $\lim_{x \to 0} g(x) = m$ (l and m are real numbers) then

(1) $\lim_{x \to a} (f(x) + g(x)) = l + m$ (Sum rule)

(2) $\lim_{x \to a} (f(x) - g(x)) = l - m$ (Difference rule)

(3) $\lim_{x \to a} (f(x).g(x)) = l.m$ (Product rule)

(4) $\lim_{x \to a} k \ f(x) = k.l$ (Constant multiple rule)

(5) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}, m \neq 0$ (Quotient rule)

(6) If $\lim_{x \to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \to a} \frac{1}{f(x)} = 0$

(7) $\lim_{x \to a} \log\{f(x)\} = \log\{\lim_{x \to a} f(x)\}$

(8) If $f(x) \leq g(x)$ for all x, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

(9) $\lim_{x \to a} [f(x)]^{g(x)} = \{\lim_{x \to a} f(x)\}^{\lim_{x \to a} g(x)}$

(10) If p and q are integers, then $\lim_{x \to a} (f(x))^{p/q} = l^{p/q}$,

provided $(l)^{p/q}$ is a real number.

If $\lim_{x \to a} f(g(x)) = f(\lim_{x \to a} g(x)) = f(m)$ provided 'f' is continuous at $g(x) = m$. e.g. $\lim_{x \to a} \ln[f(x)] = \ln(l)$, only if $l > 0$.

Some Important Expansions

In finding limits, use of expansions of following functions are useful :

(1) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

(2) $a^x = 1 + x\log a + \frac{(x\log a)^2}{2!} + ...$

(3) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(4) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...,|x| < 1$

(5) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ....,$ where $|x| < 1$

(6) $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)} = e^{1-\frac{x}{2}+\frac{x^2}{3}}\,\ldots\ldots\, = e\left(1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\ldots\right)$

(7) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$

(8) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

(9) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots\ldots$

(10) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$

(11) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$

(12) $\tanh x = x - \frac{x^3}{3} + 2x^5 - .....$

(13) $\sin^{-1} x = x + 1^2.\frac{x^3}{3!} + 3^2.1^2.\frac{x^5}{5!} + ...$

(14) $\cos^{-1} x = \left(\frac{\pi}{2}\right) - \sin^{-1} x$

(15) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(1) **Algebraic limits :** Let $f(x)$ be an algebraic function and 'a' be a real number.

Then $\lim_{x \to a} f(x)$ is known as an algebraic limit.

(i) **Direct substitution method :** If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

(ii) **Factorisation method :** In this method, numerator and denominator are factorised. The common factors are cancelled and the rest outputs the results.

(iii) **Rationalisation method :** Rationalisation is followed when we have fractional powers (like $\frac{1}{2}, \frac{1}{3}$ etc.) on expressions in numerator or denominator or in both.

After rationalisation the terms are factorised which on cancellation gives the result.

(iv) **Based on the form when $x \to \infty$:** In this case expression should be expressed as a function $1/x$ and then after removing indeterminate form, (if it is there) replace $\frac{1}{x}$ by 0.

Step I : Write down the expression in the form of rational function, *i.e.*, $\frac{f(x)}{g(x)}$, if it is not so.

Step II : If k is the highest power of x in numerator and denominator both, then divide each term of numerator and denominator by x^k.

Step III : Use the result $\lim_{x \to \infty} \frac{1}{x^n} = 0$, where $n > 0$.

An important result :

If m, n are positive integers and $a_0, b_0 \neq 0$ are non-zero real numbers, then

$$\lim_{x \to \infty} \frac{a_0 x^m + a_1 x^{m-1} + \ldots. + a_{m-1} x + a_m}{b_0 x^n + b_1 x^{n-1} + \ldots.. + b_{n-1} x + b_n} = \begin{cases} \frac{a_0}{b_0}, \text{ if } m = n \\ 0, \text{ if } m < n \\ \infty, \text{if } m > n \end{cases}$$

$$\lim_{x \to 1} (3x^2 + 4x + 5) =$$

(a) 12

(b) –1

(c) Does not exist

(d) None of these

$$\lim_{x \to 1} (3x^2 + 4x + 5) = 3(1)^2 + 4(1) + 5 = 12\,.$$

The value of $\lim_{x \to 2} \frac{3^{x/2} - 3}{3^x - 9}$ is **[MP PET 2000]**

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{1}{6}$

(d) $\ln 3$

$$\lim_{x\to 2} \frac{3^{x/2}-3}{(3^{x/2})^2-(3)^2} = \lim_{x\to 2} \frac{(3^{x/2}-3)}{(3^{x/2}-3)(3^{x/2}+3)} = \frac{1}{6}.$$

The value of $\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$ is[Rajasthan PET 1989, 92]

(a) 0

(b) na^{n-1}

(c) na^n

(d) 1

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + .. + a^{n-1})}{(x-a)}$$

$$= \lim_{x \to a} (x^{n-1} + x^{n-2}a + ... + a^{n-1}) = n \,.\, a^{n-1}.$$

$\lim_{h \to 0} \frac{1}{h}\left[\frac{1}{x+h} - \frac{1}{x}\right]$ equals [Rajasthan PET 1987]

(a) $\frac{1}{2x}$

(b) $-\frac{1}{2x}$

(c) $\frac{1}{x^2}$

(d) $-\frac{1}{x^2}$

$$\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{x+h}-\frac{1}{x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{x-(x+h)}{(x+h)x}\right]=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h}{(x+h)x}\right]=-\frac{1}{x^2}.$$

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The value of $\lim_{x \to 0} \frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2}$ is [MP PET 1999]

(a) 1

(b) –1

(c) – 2

(d) 0

$$\lim_{x \to 0} \frac{\left(\sqrt{1-x^2}-\sqrt{1+x^2}\right)}{x^2} \frac{\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}{\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)}$$

$$= \lim_{x \to 0} \frac{(1-x^2)-(1+x^2)}{x^2\left(\sqrt{1-x^2}+\sqrt{1+x^2}\right)} = \frac{-2}{2} = -1.$$

$$\lim_{x \to 3} \frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}$$ equals [UPSEAT 1991]

(a) 1

(b) $\frac{3}{2}$

(c) $\frac{1}{4}$

(d) None of these

$$\lim_{x\to 3}\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}}=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{\left(\sqrt{x-2}\right)^2-\left(\sqrt{4-x}\right)^2}$$

$$=\lim_{x\to 3}\frac{(x-3)\left(\sqrt{x-2}+\sqrt{4-x}\right)}{(2x-6)}=\lim_{x\to 3}\frac{\sqrt{x-2}+\sqrt{4-x}}{2}=\frac{1+1}{2}=1$$

$$\lim_{x \to \infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} =$$

(a) $\frac{b}{e}$

(b) $\frac{c}{f}$

(c) $\frac{a}{d}$

(d) $\frac{d}{a}$

Here the expression assumes the form $\frac{\infty}{\infty}$. We note that the highest power of x in both the numerator and denominator is 2. So we divide each terms in both the numerator and denominator by x^2.

$$\lim_{x\to\infty} \frac{ax^2 + bx + c}{dx^2 + ex + f} = \lim_{x\to\infty} \frac{a + \frac{b}{x} + \frac{c}{x^2}}{d + \frac{e}{x} + \frac{f}{x^2}} = \frac{a+0+0}{d+0+0} = \frac{a}{d}.$$

$$\lim_{x \to \infty} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right]$$ is equal to

(a) 0

(b) $\frac{1}{2}$

(c) $\log 2$

(d) e^4

$$\lim_{x\to\infty}\left[\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right] = \lim_{x\to\infty}\frac{x+\sqrt{x+\sqrt{x}}-x}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}$$

$$= \lim_{x\to\infty}\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}} = \lim_{x\to\infty}\frac{\sqrt{1+x^{-1/2}}}{\sqrt{1+\sqrt{x^{-1}+x^{-3/2}}}+1} = \frac{1}{2}.$$

The values of constants a and b so that $\lim_{x \to \infty} \left(\frac{x^2+1}{x+1} - ax - b \right) = 0$ is

(a) $a = 0, b = 0$

(b) $a = 1, b = -1$

(c) $a = -1, b = 1$

(d) $a = 2, b = -1$

We have $\lim_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$

$$\Rightarrow \lim_{x\to\infty}\frac{x^2(1-a)-x(a+b)+1-b}{x+1}=0$$

Since the limit of the given expression is zero, therefore degree of the polynomial in numerator must be less than that of denominator. As the denominator is a first degree polynomial. So, numerator must be a constant *i.e.*, a zero degree polynomial.

$\therefore\ 1-a=0$ and $a+b=0 \Rightarrow a=1$ and $b=-1$. Hence, $a=1$ and $b=-1$.

$$\lim_{x \to 1} x^x =$$

(a) 1

(b) ∞

(c) Not defined

(d) None of these

$$\lim_{x\to 1} x^{x} = \left(\lim_{x\to 1} x\right)^{\lim_{x\to 1} x} = 1^{1} = 1$$

$$\lim_{x \to 1} (1 + x)^{1/x} =$$

(a) 2 (b) e

(c) Not defined (d) None of these

$$\lim_{x \to 1} (1+x)^{1/x} = \left(\lim_{x \to 1} (1+x) \right)^{\lim_{x \to 1} \left(\frac{1}{x} \right)} = 2$$

The value of the limit of $\dfrac{x^3 - x^2 - 18}{x - 3}$ as x tends to 3 is

(a) 3 (b) 9

(c) 18 (d) 21

Let $y = \lim_{x \to 3} \frac{x^3 - x^2 - 18}{x - 3} = \lim_{x \to 3} (x^2 + 2x + 6) = 9 + 6 + 6 = 21$

The value of the limit of $\frac{x^3-8}{(x^2-4)}$ as x tends to 2 is

(a) 3

(b) $\frac{3}{2}$

(c) 1

(d) 0

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$$\lim_{x\to 2}\frac{x^3-8}{x^2-4}=\lim_{x\to 2}\frac{(x^2+2x+4)(x-2)}{(x+2)(x-2)}=\lim_{x\to 2}\frac{x^2+2x+4}{x+2}$$

$$=\frac{4+4+4}{2+2}=3\,.$$

$\lim_{x \to 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}}$ is equal to **[Rajasthan PET 1988]**

(a) $\frac{1}{2}$ (b) 2

(c) 1 (d) 0

$$\lim_{x\to 0}\left(\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\right)=\lim_{x\to 0}\left(\frac{x}{\sqrt{1+x}-\sqrt{1-x}}\times\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$

$$=\lim_{x\to 0}\left(\frac{x\left(\sqrt{1+x}+\sqrt{1-x}\right)}{1+x-1+x}\right)=\lim_{x\to 0}\left(\frac{\left(\sqrt{1+x}+\sqrt{1-x}\right)}{2}\right)=\frac{2}{2}=1$$

$\lim_{x \to a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$ equals [IIT 1978; Kurukshetra CEE 1998]

(a) $\frac{2a}{3\sqrt{3}}$

(b) $\frac{2}{3\sqrt{3}}$

(c) 0

(d) None of these

$$\lim_{x\to a}\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}=\lim_{x\to a}\left(\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right)\times\left(\frac{\sqrt{a+2x}+\sqrt{3x}}{\sqrt{a+2x}+\sqrt{3x}}\right)\times\left(\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}\right)$$

$$=\lim_{x\to a}\left\{\frac{\sqrt{3a+x}+2\sqrt{x}}{3(\sqrt{a+2x}+\sqrt{3x})}\right\}=\frac{2}{3\sqrt{3}}$$

$$\lim_{n\to\infty} \frac{1^{99} + 2^{99} + 3^{99} + ... + n^{99}}{n^{100}} =$$

[EAMCET 1994]

(a) $\frac{99}{100}$

(b) $\frac{1}{100}$

(c) $\frac{1}{99}$

(d) $\frac{1}{101}$

$$\lim_{n\to\infty} \frac{1^{99}+2^{99}+3^{99}+....+n^{99}}{n^{100}} = \lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r^{99}}{n^{100}} \right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n} \right)^{99} = \int_0^1 x^{99}\, dx = \left[\frac{x^{100}}{100} \right]_0^1 = \frac{1}{100}.$$

The values of constants 'a' and 'b' so that

$$\lim_{x \to \infty} \left(\frac{x^2 - 1}{x + 1} - ax - b \right) = 2 \text{ is}$$

(a) $a = 0, b = 0$

(b) $a = 1, b = -1$

(c) $a = 1, b = -3$

(d) $a = 2, b = -1$

$$\lim_{x\to\infty}\left(\frac{x^2-1}{x+1}-ax-b\right)=2 \Rightarrow \lim_{x\to\infty} x-1-ax-b=2 \Rightarrow \lim_{x\to\infty} x(1-a)-(1+b)=2\,.$$

Comparing the coefficient of both sides, $1-a=0$ and $1+b=-2 \Rightarrow a=1, b=-3$

$$\lim_{n\to\infty}\left[\frac{\sum n^2}{n^3}\right] =$$ [Rajasthan PET 1999, 2002]

(a) $-\frac{1}{6}$ (b) $\frac{1}{6}$

(c) $\frac{1}{3}$ (d) $\frac{-1}{3}$

$$\lim_{n\to\infty}\left[\frac{n(n+1)(2n+1)}{6n^3}\right]=\lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}=\frac{1}{3}$$

Note : ❒ Students should remember that,

$$\lim_{n\to\infty}\frac{\sum n}{n^2}=\frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty}\frac{\sum n^2}{n^3}=\frac{1}{3}.$$

$$\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+\ldots\ldots+\frac{n}{1-n^2}\right]$$ is equal to [IIT 1984; DCE 2000]

(a) 0

(b) $-\frac{1}{2}$

(c) $\frac{1}{2}$

(d) None of these

$$\lim_{n\to\infty}\left[\frac{1}{1-n^2}+\frac{2}{1-n^2}+\ldots\ldots+\frac{n}{1-n^2}\right]=\lim_{n\to\infty}\frac{\sum n}{1-n^2}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2+n}{1-n^2}=-\frac{1}{2}.$$

If $f(x)=\frac{2}{x-3}, g(x)=\frac{x-3}{x+4}$ and

$h(x)=-\frac{2(2x+1)}{x^2+x-12}$ then $\lim_{x\to 3}[f(x)+g(x)+h(x)]$ is

(a) -2

(b) -1

(c) $-\frac{2}{7}$

(d) 0

We have $f(x)+g(x)+h(x)=\dfrac{x^2-4x+17-4x-2}{x^2+x-12}=\dfrac{x^2-8x+15}{x^2+x-12}=\dfrac{(x-3)(x-5)}{(x-3)(x+4)}$

$$\therefore \lim_{x\to 3}[f(x)+g(x)+h(x)]=\lim_{x\to 3}\frac{(x-3)(x-5)}{(x-3)(x+4)}=-\frac{2}{7}$$

If $\lim_{x\to\infty}\left[\frac{x^3+1}{x^2+1}-(ax+b)\right]=2$, then **[Karnataka CET 2000]**

(a) $a=1$ and $b=1$

(b) $a=1$ and $b=-1$

(c) $a=1$ and $b=-2$

(d) $a=1$ and $b=2$

$$\lim_{x\to\infty}\left(\frac{x^3+1}{x^2+1}-(ax+b)\right)=2 \Rightarrow \lim_{x\to\infty}\left(\frac{x^3(1-a)-bx^2-ax+(1-b)}{x^2+1}\right)=2$$

$$\Rightarrow \lim_{x\to\infty}[x^3(1-a)-bx^2-ax+(1-b)]=2(x^2+1).$$

Comparing the coefficients of both sides, $1-a=0$ and $-b=2$ or $a=1, b=-2$.

$$\lim_{x \to \infty} \frac{(x+1)^{10} + (x+2)^{10} + \ldots.. + (x+100)^{10}}{x^{10} + 10^{10}}$$ is equal to[**AMU 2000**]

(a) 0 (b) 1

(c) 10 (d) 100

$$\lim_{x\to\infty}\frac{(x+1)^{10}+(x+2)^{10}+\ldots\ldots+(x+100)^{10}}{x^{10}+10^{10}}$$

$$=\lim_{x\to\infty}\frac{x^{10}\left[\left(1+\frac{1}{x}\right)^{10}+\left(1+\frac{2}{x}\right)^{10}+\ldots+\left(1+\frac{100}{x}\right)^{10}\right]}{x^{10}\left[1+\frac{10^{10}}{x^{10}}\right]}=100\ .$$

Let $f(x) = 4$ and $f'(x) = 4$, then

$$\lim_{x \to 2} \frac{xf(2) - 2f(x)}{x - 2}$$ equals **[Rajasthan 2000; AIEEE 2002]**

(a) 2

(b) – 2

(c) – 4

(d) 3

$$y = \lim_{x\to 2} \frac{xf(2) - 2f(x)}{x-2}$$

$$\Rightarrow y = \lim_{x\to 2} \frac{xf(2) - 2f(2) + 2f(2) - 2f(x)}{x-2}$$

$$\Rightarrow y = \lim_{x\to 2} \frac{-2f(x) + 2f(2) + xf(2) - 2f(2)}{(x-2)}$$

$$\Rightarrow y = \lim_{x\to 2} -2\frac{[f(x) - f(2)]}{x-2} + \lim_{x\to 2} \frac{f(2).(x-2)}{(x-2)}$$

$$\Rightarrow y = -2\lim_{x\to 2} \frac{f(x) - f(2)}{x-2} + f(2)$$

$$\Rightarrow y = -2 \lim_{x\to 2} f'(x) + f(2) = -8 + 4 = -4$$

(1) Trigonometric limits :

To evaluate trigonometric limits the following results are very important.

(i) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1 = \lim\limits_{x \to 0} \dfrac{x}{\sin x}$

(ii) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1 = \lim\limits_{x \to 0} \dfrac{x}{\tan x}$

(iii) $\lim\limits_{x \to 0} \dfrac{\sin^{-1} x}{x} = 1 = \lim\limits_{x \to 0} \dfrac{x}{\sin^{-1} x}$

(iv) $\lim\limits_{x \to 0} \dfrac{\tan^{-1} x}{x} = 1 = \lim\limits_{x \to 0} \dfrac{x}{\tan^{-1} x}$

(v) $\lim\limits_{x \to 0} \dfrac{\sin x^{0}}{x} = \dfrac{\pi}{180}$

(vi) $\lim_{x \to 0} \cos x = 1$

(vii) $\lim_{x \to a} \frac{\sin(x - a)}{x - a} = 1$

(viii) $\lim_{x \to a} \frac{\tan(x - a)}{x - a} = 1$

(ix) $\lim_{x \to a} \sin^{-1} x = \sin^{-1} a, |a| \le 1$

(x) $\lim_{x \to a} \cos^{-1} x = \cos^{-1} a; |a| \le 1$

(xi) $\lim_{x \to a} \tan^{-1} x = \tan^{-1} a; -\infty < a < \infty$

(xii) $\lim_{x \to \infty} \frac{\sin x}{x} = \lim_{x \to \infty} \frac{\cos x}{x} = 0$

(xiii) $\lim_{x \to \infty} \frac{\sin(1/x)}{(1/x)} = 1$

$\lim_{x \to 1} (1 - x) \tan\left(\frac{\pi x}{2}\right) =$ [IIT 1978, 84; Rajasthan PET 1997, 2001; UPSEAT 2003]

(a) $\frac{\pi}{2}$

(b) π

(c) $\frac{2}{\pi}$

(d) 0

$\lim_{x \to 1} (1-x)\tan\left(\frac{\pi x}{2}\right)$, Put $1-x=y \Rightarrow$ as $x \to 1, y \to 0$

Thus $\lim_{y \to 0} y \tan \frac{\pi(1-y)}{2} = \lim_{y \to 0} \frac{2}{\pi} \cdot \frac{\left(\frac{\pi y}{2}\right)}{\tan\left(\frac{\pi y}{2}\right)} = \frac{2}{\pi} \times 1 = \frac{2}{\pi}$.

$$\lim_{x \to 1} \frac{\sqrt{1-\cos 2(x-1)}}{x-1}$$ [IIT 1998; UPSEAT 2001]

(a) Exists and it equal $\sqrt{2}$

(b) Exists and it equals $-\sqrt{2}$

(c) Does not exist because $x-1 \to 0$

(d) Does not exist because left hand limit is not equal to right hand limit

$$f(1+) = \lim_{h\to 0} f(1+h) = = \lim_{h\to 0} \frac{\sqrt{1-\cos 2h}}{h} = \lim_{h\to 0} \sqrt{2}\,\frac{\sin h}{h} = \sqrt{2}$$

$$f(1-) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \frac{\sqrt{1-\cos(-2h)}}{-h} = \lim_{h\to 0} \sqrt{2}\,\frac{\sin h}{-h} = -\sqrt{2}.$$

$\therefore$ limit does not exist because left hand limit is not equal to right hand limit.

$$\lim_{x \to 0} \frac{(1 - \cos 2x) \sin 5x}{x^2 \sin 3x} =$$ **[MP PET 2000; UPSEAT 2000; Karmataka CET 2002]**

(a) $\frac{10}{3}$ (b) $\frac{3}{10}$

(c) $\frac{6}{5}$ (d) $\frac{5}{6}$

$$\lim_{x \to 0} \frac{2\sin^2 x \quad \sin 5x \quad 3x \quad 5x}{x^2 \quad \sin 3x \quad 3x \quad 5x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x}{x^2} \cdot \frac{3x}{\sin 3x} \cdot \frac{\sin 5x}{5x} \cdot \frac{5x}{3x} = 2 \cdot \frac{5}{3} = \frac{10}{3} .$$

$$\lim_{x \to 0} \frac{x^3}{\sin x^2} =$$

(a) 0

(b) $\frac{1}{3}$

(c) 3

(d) $\frac{1}{2}$

$$\lim_{x\to 0}\frac{x^3}{\sin x^2}=\lim_{x\to 0}\frac{x^2}{\sin x^2}\cdot x$$

$$=\left(\lim_{x\to 0}\frac{x^2}{\sin x^2}\right)\left(\lim_{x\to 0} x\right) = 1.0 = 0.$$

$$\lim_{x \to 0} \frac{\sin 3x + \sin x}{x} =$$

(a) $\frac{1}{3}$

(b) 3

(c) 4

(d) $\frac{1}{4}$

$$\lim_{x \to 0} \frac{\sin 3x + \sin x}{x} = \lim_{x \to 0} \frac{\sin 3x}{x} + \lim_{x \to 0} \frac{\sin x}{x}$$

$$= \lim_{x \to 0} \frac{\sin 3x}{3x}.3 + \lim_{x \to 0} \frac{\sin x}{x} = 1.3 + 1 = 4.$$

If $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$, then $\lim_{x\to 0} f(x) =$ **[IIT 1988; UPSEAT 1988; SCRA 1996]**

(a) 1

(b) 0

(c) –1

(d) None of these

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0} \sin\frac{1}{x}\right)$$

= 0 × (A number oscillating between – 1 and 1) = 0.

If $f(x)=\begin{cases}\dfrac{\sin[x]}{[x]}, & [x]\neq 0\\ 0\quad, & [x]=0\end{cases}$, then $\lim\limits_{x\to 0} f(x)$ equals **[IIT 1985; Rajasthan PET 1995]**

(a) 1 (b) 0

(c) –1 (d) Does not exist

In closed interval of $x = 0$ at right hand side $[x] = 0$ and at left hand side $[x] = -1$. Also $[0] = 0$.

Therefore function is defined as $$f(x) = \begin{cases} \dfrac{\sin[x]}{[x]}, & (-1 \le x < 0) \\ 0, & (0 \le x < 1) \end{cases}$$

$$\therefore \text{ Left hand limit } = \lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{\sin[x]}{[x]} = \frac{\sin(-1)}{-1} = \sin 1^c$$

Right hand limit = 0, Hence, limit doesn't exist.

$\lim_{x \to 0} \frac{\tan x - \sin x}{x^3}$ [IIT 1974; Rajasthan PET 2000]

(a) $\frac{1}{2}$

(b) $-\frac{1}{2}$

(c) $\frac{2}{3}$

(d) None of these

$$\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x-\sin x\cos x}{x^3\cos x}$$

$$=\lim_{x\to 0}\frac{\sin x\left(2\sin^2\frac{x}{2}\right)}{x^3\cos x}=\lim_{x\to 0}\left[\frac{\sin x}{x}\cdot\frac{2}{\cos x}\cdot\frac{\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{1}{4}\right]=\frac{1}{2}$$

$$\lim_{x \to \pi/2} \frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} =$$ **[Kerala (Engg.) 2001]**

(a) $\log a$

(b) $\log 2$

(c) a

(d) $\log x$

$$\lim_{x \to \pi/2} \left(\frac{a^{\cot x} - a^{\cos x}}{\cot x - \cos x} \right) = \lim_{x \to \pi/2} a^{\cos x} \left(\frac{a^{\cot x - \cos x} - 1}{\cot x - \cos x} \right)$$

$$= a^{\cos(\pi/2)} \lim_{x \to \pi/2} \left(\frac{a^{\cot x - \cos x} - 1}{\cot x - \cos x} \right) = 1 \log a = \log a$$

If $\lim_{x \to 0} \frac{[(a-n)nx - \tan x]\sin nx}{x^2} = 0$, where n is non-zero real number, then a is equal to **[IIT Screening 2003]**

(a) 0

(b) $\frac{n+1}{n}$

(c) n

(d) $n + \frac{1}{n}$

$$\lim_{x\to 0} n\frac{\sin nx}{nx}.\lim_{x\to 0}\left((a-n)n-\frac{\tan x}{x}\right)=0$$

$$\Rightarrow n[(a-n)n-1]=0 \Rightarrow (a-n)n=1 \Rightarrow a=n+\frac{1}{n}.$$

(ii) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(iii) $\lim_{x \to e} \log_e x = 1$

(iv) $\lim_{x \to 0} \frac{\log(1-x)}{x} = -1$

(v) $\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \log_a e, a > 0, \neq 1$

$$\lim_{x \to a} \frac{\log\{1+(x-a)\}}{(x-a)} =$$

(a) -1

(b) 2

(c) 1

(d) -2

Let $x - a = y$, when $x \to a,\ y \to 0,$

$$\therefore \quad \text{The given limit} = \lim_{y \to 0} \frac{\log\{1+y\}}{y} = 1 .$$

$$\lim_{h \to 0} \frac{\log_{10}(1+h)}{h} =$$

(a) 1

(b) $\log_{10} e$

(c) $\log_e 10$

(d) None of these

$$\lim_{h\to 0} \frac{\log_e(1+h)}{h} \cdot \frac{1}{\log_e 10} = \log_{10} e \, .$$

If $\lim_{x \to 0} \dfrac{\log(3+x) - \log(3-x)}{x} = k$, then the value of k is **[AIEEE 2003]**

(a) 0

(b) $-\dfrac{1}{3}$

(c) $\dfrac{2}{3}$

(d) $-\dfrac{2}{3}$

$$\lim_{x \to 0} \frac{\log(3+x) - \log(3-x)}{x} = \lim_{x \to 0} \frac{\log\left(\frac{3+x}{3-x}\right)}{x} = \lim_{x \to 0} \frac{\log\left(\frac{1+(x/3)}{1-(x/3)}\right)}{x}$$

$$= \lim_{x \to 0} \frac{\log(1+(x/3))}{x} - \lim_{x \to 0} \frac{\log(1-(x/3))}{x} = \frac{1}{3} - \left(-\frac{1}{3}\right) = \frac{2}{3}.$$

(a) $$\lim_{x \to 0} \frac{e^{x} - 1}{x} = 1$$

(b) $$\lim_{x \to 0} \frac{a^{x} - 1}{x} = \log_e a$$

(c) $$\lim_{x \to 0} \frac{e^{\lambda x} - 1}{x} = \lambda \qquad (\lambda \neq 0)$$

Based on the form 1^{∞}: To evaluate the exponential form 1^{∞} we use the following results.

(a) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$,

then $\lim_{x\to a} \{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a} \frac{f(x)}{g(x)}}$, or

when $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$.

Then $\lim_{x\to a} \{f(x)\}^{g(x)} = \lim_{x\to a} [1 + f(x) - 1]^{g(x)} = \boxed{e^{\lim_{x\to a} (f(x)-1)g(x)}}$

(b) $\lim_{x \to 0} (1 + x)^{1/x} = e$

(c) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e$

(d) $\lim_{x \to 0} (1 + \lambda x)^{1/x} = e^{\lambda}$

(e) $\lim_{x \to \infty} \left(1 + \frac{\lambda}{x}\right)^{x} = e^{\lambda}$

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$$\lim_{x \to \infty} a^x = \begin{cases} \infty, \text{ if } a > 1 \\ 0, \text{ if } a < 1 \end{cases}$$

i.e., $a^{\infty} = \infty$, if $a > 1$ and $a^{\infty} = 0$ if $a < 1$.

$$\lim_{x\to 0}\frac{e^{\alpha x}-e^{\beta x}}{x}=$$ [MP PET 1994]

(a) $\alpha+\beta$

(b) $\frac{1}{\alpha}+\beta$

(c) $\alpha^2-\beta^2$

(d) $\alpha-\beta$

$$\lim_{x\to 0}\frac{e^{\alpha x}-e^{\beta x}}{x}=\lim_{x\to 0}\frac{(e^{\alpha x}-1)-(e^{\beta x}-1)}{x}$$

$$=\lim_{x\to 0}\frac{e^{\alpha x}-1}{x}-\lim_{x\to 0}\frac{e^{\beta x}-1}{x}=\alpha-\beta .$$

The value of $\lim_{x \to \infty} \left(\frac{x+3}{x+1} \right)^{x+2}$ is **[UPSEAT 2003]**

(a) e^4

(b) 0

(c) 1

(d) e^2

$$\lim_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2}=\lim_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}.(x+2).\frac{2}{(x+1)}}$$

$$=\lim_{x\to\infty}\left(\left(1+\frac{2}{x+1}\right)^{\frac{x+1}{2}}\right)^{2.\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)}=e^{2\lim\limits_{x\to\infty}\left[\left(1+\frac{2}{x}\right)/\left(1+\frac{1}{x}\right)\right]}=e^{2}.$$

Alternative method : $\lim\limits_{x\to\infty}\left(\frac{x+3}{x+1}\right)^{x+2}=\lim\limits_{x\to\infty}\left(1+\frac{2}{x+1}\right)^{x+2}=e^{\lim\limits_{x\to\infty}\frac{2}{x+1}(x+2)}=e^{\lim\limits_{x\to\infty}2\left(\frac{1+\frac{2}{x}}{1+\frac{1}{x}}\right)}=e^{2}$

If a, b, c, d are positive, then $\lim_{x \to \infty} \left(1 + \frac{1}{a+bx}\right)^{c+dx}$ **[EAMCET 1992]**

(a) $e^{d/b}$ (b) $e^{c/a}$

(c) $e^{(c+d)/(a+b)}$ (d) e

$$\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx} = \lim_{x\to\infty}\left\{\left(1+\frac{1}{a+bx}\right)^{a+bx}\right\}^{\frac{c+dx}{a+bx}}$$

$$= e^{d/b} \quad \left\{\because \lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{a+bx} = e \text{ and } \lim_{x\to\infty}\frac{c+dx}{a+bx} = \frac{d}{b}\right\}$$

Alternative method : $e^{\lim\limits_{x\to\infty}\left(\frac{1}{a+bx}\right)\left(\frac{c+dx}{1}\right)} = e^{d/b}$.

$\lim_{x \to 0} x^x =$ **[Roorkee 1987]**

(a) 0

(b) 1

(c) e

(d) None of these

Let $y = x^x \Rightarrow \log y = x \log x$;

$$\therefore \lim_{y \to 0} \log y = \lim_{x \to 0} x \log x = 0 = \log 1 \Rightarrow \lim_{x \to 0} x^x = 1$$

$$\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)} =$$ [AMU 2002]

(a) e

(b) e^2

(c) e^{-1}

(d) 1

$$\lim_{n\to\infty}\left(\frac{n^2-n+1}{n^2-n-1}\right)^{n(n-1)} = \lim_{n\to\infty}\left(\frac{n(n-1)+1}{n(n-1)-1}\right)^{n(n-1)}$$

$$= \lim_{n\to\infty}\frac{\left(1+\dfrac{1}{n(n-1)}\right)^{n(n-1)}}{\left(1-\dfrac{1}{n(n-1)}\right)^{n(n-1)}} = \frac{e}{e^{-1}} = e^2 .$$

Alternative Method: $\lim_{n\to\infty}\left(1+\frac{2}{n^2-n-1}\right)^{n(n-1)} = e^{\lim_{n\to\infty}\frac{2n(n-1)}{n^2-n-1}} = e^2 .$

(5) **L' Hospital's rule :** If $f(x)$ and $g(x)$ be two functions of x such that

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

(ii) Both are continuous at $x = a$

(iii) Both are differentiable at $x = a$.

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided that $g'(a) \neq 0$

- The above rule is also applicable if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$.

- If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x), g'(x)$ satisfy all the condition embodied in L' Hospital rule, we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get, $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$. Sometimes it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

$$\lim_{x \to 0} \frac{1-\cos mx}{1-\cos nx} =$$ **[Kerala (Engg.) 2002]**

(a) m/n (b) n/m

(c) $\frac{m^2}{n^2}$ (d) $\frac{n^2}{m^2}$

$$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\lim_{x\to 0}\left\{\frac{2\sin^2\frac{mx}{2}}{2\sin^2\frac{nx}{2}}\right\}=\lim_{x\to 0}\left[\left\{\frac{\sin\frac{mx}{2}}{\frac{mx}{2}}\right\}^2\frac{m^2x^2}{4}\cdot\frac{1}{\left\{\frac{\sin\frac{nx}{2}}{\frac{nx}{2}}\right\}^2}\cdot\frac{4}{n^2x^2}\right]=\frac{m^2}{n^2}\times 1=\frac{m^2}{n^2}$$

Trick : Apply L-Hospital rule ,

$$\lim_{x\to 0}\frac{1-\cos mx}{1-\cos nx}=\lim_{x\to 0}\frac{m\sin mx}{n\sin nx}=\lim_{x\to 0}\frac{m^2\cos mx}{n^2\cos nx}=\frac{m^2}{n^2}.$$

$$\lim_{\alpha \to \pi/4} \frac{\sin\alpha - \cos\alpha}{\alpha - \pi/4} =$$ [IIT Screening 1997; AMU 1997]

(a) $\sqrt{2}$

(b) $1/\sqrt{2}$

(c) 1

(d) None of these

$$\lim_{\alpha \to \pi/4} \frac{\sin\alpha - \cos\alpha}{\alpha - \pi/4} \left(\frac{0}{0} \text{ form}\right) = \lim_{\alpha \to \pi/4} \frac{\cos\alpha + \sin\alpha}{1} \quad \text{(By 'L' Hospital rule)}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

$$\lim_{x \to a} \frac{x^3 - a^3}{x^2 - a^2} =$$

(a) 0 (b) Not defined

(c) $2a$ (d) $\frac{3a}{2}$

$$\lim_{x \to a} \frac{x^3 - a^3}{x^2 - a^2} \quad \left(\frac{0}{0} \text{ form}\right) = \lim_{x \to a} \frac{3x^2}{2x} \quad (\text{By 'L' Hospital rule}) = \frac{3a^2}{2a} = \frac{3a}{2}.$$

$$\lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$$ [Roorkee 1983]

(a) $1/2\sqrt{x}$

(b) $1/2\sqrt{h}$

(c) Zero

(d) None of these

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

Trick : Applying 'L' Hospital's rule, [Differentiating Nr and Dr with respect to h]

We get, $\lim_{h\to 0}\dfrac{\dfrac{1}{2\sqrt{x+h}}-0}{1} = \dfrac{1}{2\sqrt{x}}$.

$$\lim_{\alpha \to \beta} \frac{\sin^2 \alpha - \sin^2 \beta}{\alpha^2 - \beta^2}$$ **[MP PET 2001]**

(a) 0

(b) 1

(c) $\frac{\sin \beta}{\beta}$

(d) $\frac{\sin 2\beta}{2\beta}$

$$\lim_{\alpha\to\beta}\frac{\sin^2\alpha-\sin^2\beta}{\alpha^2-\beta^2}=\lim_{\alpha-\beta\to 0}\frac{\sin(\alpha+\beta)\sin(\alpha-\beta)}{(\alpha+\beta)\ (\alpha-\beta)}$$

$$=\lim_{\alpha-\beta\to 0}\frac{\sin(\alpha-\beta)}{(\alpha-\beta)}\lim_{\alpha-\beta\to 0}\frac{\sin(\alpha+\beta)}{(\alpha+\beta)}=\lim_{\alpha\to\beta}\frac{\sin(\alpha+\beta)}{(\alpha+\beta)}=\frac{\sin 2\beta}{2\beta}.$$

Trick : By L' Hospital's rule, $\lim_{\alpha\to\beta}\frac{2\sin\alpha\cos\alpha}{2\alpha}=\frac{\sin 2\beta}{2\beta}$.

$$\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x}$$ equals [IIT 1971]

(a) 2/3 (b) 1/3

(c) 1/2 (d) 0

$$\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x} = \lim_{x \to 0} \left\{ \frac{\frac{2 \tan 2x}{2x} - 1}{3 - \frac{\sin x}{x}} \right\} = \frac{1}{2}.$$

If $G(x) = -\sqrt{25 - x^2}$, then $\lim_{x \to 1} \frac{G(x) - G(1)}{x - 1}$ equals [IIT 1983]

(a) 1/24

(b) 1/5

(c) $-\sqrt{24}$

(d) None of these

$$\lim_{x\to 1}\frac{G(x)-G(1)}{x-1}=\lim_{x\to 1}\frac{-\sqrt{25-x^2}+\sqrt{24}}{x-1}$$

[Multiply both numerator and denominator by $(\sqrt{24}+\sqrt{25-x^2})$]

$$=\lim_{x\to 1}\frac{x+1}{\sqrt{24}+\sqrt{25-x^2}}=\frac{1}{\sqrt{24}}$$

Alternative method: By L'-Hospital rule, $\lim_{x\to 1}\frac{G'(x)}{1}=\lim_{x\to 1}\frac{-1(-2x)}{2\sqrt{25-x^2}}=\frac{1}{\sqrt{24}}$

If $f(a)=2, f'(a)=1, g(a)=1, g'(a)=2,$

then $\lim_{x \to a} \frac{g(x)f(a)-g(a)f(x)}{x-a}$ equals

[IIT 1983; Rajasthan PET 1990; MP PET 1995; DCE 1999; Karnataka CET 1999, 2003]

(a) -3

(b) $\frac{1}{3}$

(c) 3

(d) $-\frac{1}{3}$

Applying L – Hospital's rule, we get,

$$\lim_{x \to a} \frac{g(x)\, f(a) - g(a)\, f(x)}{x - a} = \lim_{x \to a} \frac{g'(x)\, f(a) - g(a)\, f'(x)}{1}$$

$$= g'(a)\, f(a) - g(a)\, f'(a) = 2 \times 2 - 1 \times (1) = 3.$$

$$\lim_{x \to 0} \frac{(1+x)^n - 1}{x} =$$ [**Kurukshetra CEE 2002**]

(a) n

(b) 1

(c) -1

(d) None of these

$$\lim_{x \to 0} \frac{(1 + nx + {}^nC_2x^2 + \ldots\ldots \text{ higher powers of } x \text{ to } x^n) - 1}{x} = n$$

Trick : Apply L- Hospital rule.

$\lim_{x \to 0} \frac{\sin x + \log(1 - x)}{x^2}$ is equal to [Roorkee 1995]

(a) 0

(b) $\frac{1}{2}$

(c) $-\frac{1}{2}$

(d) None of these

Apply L- Hospital rule, we get, $\lim_{x \to 0} \frac{\cos x - \frac{1}{1-x}}{2x} = \lim_{x \to 0} \frac{-\sin x - \frac{1}{(1-x)^2}}{2} = -\frac{1}{2}$

Alternative method :

$$\lim_{x \to 0} \frac{\sin x + \log(1-x)}{x^2} = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\right)}{x^2} + \lim_{x \to 0} \frac{\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\right)}{x^2}$$

$$\left(\because \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots \text{ and } \log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} \ldots\ldots\ldots\right)$$

Hence, $\lim_{x \to 0} \frac{\frac{-x^2}{2} - x^3\left(\frac{1}{3!} + \frac{1}{3}\right) - \frac{x^4}{4} \ldots.}{x^2} = -\frac{1}{2}.$

$\lim_{x \to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$ is equal to **[Rajasthan PET 2000]**

(a) 0

(b) 1

(c) – 1

(d) $\frac{1}{2}$

$$\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}\quad\left(\frac{0}{0}\text{ form}\right)$$

Applying L-Hospital's rule,

$$=\lim_{x\to 0}\frac{\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{1+x^2}}{3x^2}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\dfrac{-1}{2}\times\dfrac{-2x}{(1-x^2)^{3/2}}+\dfrac{2x}{(1+x^2)^2}}{6x}=\lim_{x\to 0}\frac{1}{6}\left[\frac{1}{(1-x^2)^{3/2}}+\frac{2}{(1+x^2)^2}\right]=\frac{1}{2}.$$

$$\lim_{x \to 0} \frac{4^x - 9^x}{x(4^x + 9^x)} =$$ **[EAMCET 2002]**

(a) $\log\left(\frac{2}{3}\right)$

(b) $\frac{1}{2}\log\left(\frac{3}{2}\right)$

(c) $\frac{1}{2}\log\left(\frac{3}{2}\right)$

(d) $\log\left(\frac{3}{2}\right)$

$$y = \lim_{x \to 0} \frac{4^x - 9^x}{x(4^x + 9^x)} \qquad \left(\frac{0}{0} \text{ form}\right)$$

Using L-Hospital's rule,

$$y = \lim_{x \to 0} \frac{4^x \log 4 - 9^x \log 9}{(4^x + 9^x) + x(4^x \log 4 + 9^x \log 9)}$$

$$\Rightarrow y = \frac{\log 4 - \log 9}{2} \Rightarrow y = \frac{\log\left(\frac{2}{3}\right)^2}{2} = \log \frac{2}{3}.$$

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If $f(a)=2$, $f'(a)=1$, $g(a)=-3$, $g'(a)=-1$,

then $\lim_{x\to a}\frac{f(a)g(x)-f(x)g(a)}{x-a}=$ **[Karnataka CET 2003]**

(a) 1 (b) 6

(c) – 5 (d) – 1

$$\lim_{x\to a}\frac{f(a)g(x)-f(x)g(a)}{x-a}\quad\left(\frac{0}{0}\text{ form}\right)$$

Using L-Hospital's rule, $\lim_{x\to a}\frac{f(a)g'(x)-f'(x)g(a)}{1-0}$

$$= f(a)\times g'(a)-f'(a)\times g(a) = 2\times(-1)-1\times(-3) = 1.$$

Let $f(a) = g(a) = k$ and their n^{th} derivatives $f^n(a), g^n(a)$ exist and are not equal for some *n*.

If $\lim_{x \to a} \frac{f(a)g(x) - f(a) - g(a)f(x) + g(a)}{g(x) - f(x)} = 4$, then the value of k is [AIEEE 2003]

(a) 4 (b) 2

(c) 1 (d) 0

$$\lim_{x\to a}\frac{k\,g(x)-k\,f(x)}{g(x)-f(x)}=4$$

By L-Hospital' rule, $\lim_{x\to a} k\left[\frac{g'(x)-f'(x)}{g'(x)-f'(x)}\right]=4$, $\therefore\ k=4$.

$$\lim_{x \to \pi/6} \left[\frac{3 \sin x - \sqrt{3} \cos x}{6x - \pi} \right]$$

[EAMCET 2003]

(a) $\sqrt{3}$

(b) $\frac{1}{\sqrt{3}}$

(c) $-\sqrt{3}$

(d) $-\frac{1}{\sqrt{3}}$

Using L–Hospital's rule

$$, \lim_{x\to\pi/6} \frac{3\cos x + \sqrt{3}\sin x}{6} = \frac{3.\frac{\sqrt{3}}{2} + \sqrt{3}.\frac{1}{2}}{6} = \frac{1}{\sqrt{3}}.$$

Given that $f'(2) = 6$ and $f'(1) = 4$,

then $\lim_{h \to 0} \dfrac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)} =$ **[IIT Screening 2003]**

(a) Does not exist

(b) $-\dfrac{3}{2}$

(c) $\dfrac{3}{2}$

(d) 3

$$\lim_{h \to 0} \frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)} = \lim_{h \to 0} \frac{f'(2h+2+h^2)(2+2h)}{f'(h-h^2+1)(1-2h)} = \frac{6 \times 2}{4 \times 1} = 3\,.$$