Probability

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Definition

(i) ***Sample–Space*** **:** The set of all possible outcomes of an experiment is called the **Sample–Space(s)**.

(ii) ***Event*** **:** A sub set of sample–space is called an **Event**.

(iii) ***Complement Of An Event A*** **:** The set of all out comes which are in S but not in A is called the **Complement Of The Event A denoted by** $\overline{A}$ **or** A^c.

Event Experiment

→ Tossing of a coin

→ Rolling of a dice

→ Drawing a card

$S = \{1,2,3,4,5,6\}$

$E_1 = \{3,6\}$ $\overline{E_1} = \{1,2,4,5\}$

$E_2 = \{2,4,6\}$ $\overline{E_2} = \{1,3,5\}$

	Outcome	E_1	E_2
Trial-1 :	'4'	D.N.O	Occur
Trial-2 :	'5'	D.N.O	D.N.O
Trial-3 :	'6'	Occur	Occur

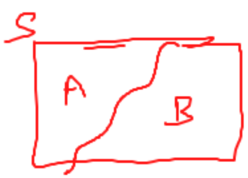
Definition

(iv) ***Mutually Exclusive Events*** **:** Two events are said to be **Mutually Exclusive** (or disjoint or incompatible) if the occurence of one precludes (rules out) the simultaneous occurence of the other . If A & B are two mutually exclusive events then P (A & B) = 0.

$A \cap B = \emptyset$ $P(A \cap B) = 0$

(v) ***Equally Likely Events*** **:** Events are said to be **Equally Likely** when each event is as likely to occur as any other event. $\hookrightarrow P(E_1) = P(E_2)$

(vi) ***Exhaustive Events*** **:** Events A,B,C .... L are said to be **Exhaustive Events** if no event outside this set can result as an outcome of an experiment . For example, if A & B are two events defined on a sample space S, then A & B are exhaustive $\Rightarrow A \cup B = S \Rightarrow P(A \cup B) = 1$.



$\boxed{A \cup B = S}$

$S = \{1, 2, 3, 4, 5, 6\}$

$E_1 = \{2, 4, 6\}$

$E_2 = \{1, 3, 5\}$

No common element

$P(E_1) = \frac{3}{6} = \frac{1}{2}$

$P(E_2) = \frac{3}{6} = \frac{1}{2}$

CLASSICAL DEF. OF PROBABILITY : If n represents the total number of equally likely, mutually exclusive and exhaustive outcomes of an experiment and m of them are favourable to the happening of the event A, then the probability of happening of the event A is given by P(A) = m/n.

(1) $0 \le P(A) \le 1$

$P(\overline{A}) = \frac{n-m}{n} = 1 - \frac{m}{n}$

(2) $P(A) + P(\overline{A}) = 1$, Where $\overline{A}$ = Not A.

(3) If x cases are favourable to A & y cases are favourable to $\overline{A}$ then $P(A) = \frac{x}{(x+y)}$ and $P(\overline{A}) = \frac{y}{(x+y)}$ We say that **ODDS IN FAVOUR OF A** are x: y & odds against A are y : x

$\frac{\text{fav outcomes}}{\text{unfav outcomes}} = \frac{m}{n-m}$ odds in favour

$\frac{\text{unfav outcomes}}{\text{fav outcomes}}$

Problems

In a random experiment, a fair die is rolled until two fours are obtained in succession. The probability that the experiment will end in the fifth throw of the die is equal to **(2019 Main, 12 Jan II)**

(a) $\frac{175}{6^5}$ (b) $\frac{225}{6^5}$

(c) $\frac{200}{6^5}$ (d) $\frac{150}{6^5}$

1,2,3,5,6

$\underbrace{\_\ \_}_{}$ $\_$ 4 4

6 6 5 1 1

$(36-1)$

Fav outcomes = $35 \times 5 \times 1 \times 1$

Total outcomes = 6^5

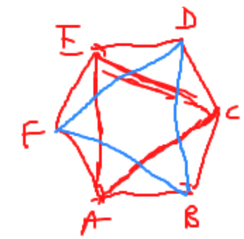
Prob = $\frac{35 \times 5}{6^5} = \frac{175}{6^5}$

Problems

If ~~there~~ three of the six vertices of a regular hexagon are chosen at random, then the probability that the triangle formed with these chosen vertices is equilateral is

(2019 Main, 12 April I)

(a) $\frac{1}{10}$ (b) $\frac{1}{5}$ (c) $\frac{3}{10}$ (d) $\frac{3}{20}$



Total Outcomes: $^6C_3$

Fav Outcomes: 2 (ACE or BDF)

Prob $= \frac{2}{^6C_3} = \frac{2\times 3\times 2\times 1}{6\times 5\times 4} = \frac{2}{20} = \frac{1}{10}$

Problems

Let $S = \{1, 2, \ldots, 20\}$. A subset B of S is said to be "nice", if the sum of the elements of B is 203. Then, the probability that a randomly chosen subset of S is "nice", is **(2019 Main, 11 Jan II)**

(a) $\frac{6}{2^{20}}$ (b) $\frac{4}{2^{20}}$ (c) $\frac{7}{2^{20}}$ (d) $\frac{5}{2^{20}}$

Total Subsets of S $= 2^{20}$

$\Rightarrow$ total outcomes $= 2^{20}$

Sum of all elements $= 1+2+3 \text{ — } 20$

$= \frac{20(20+1)}{2} = 210$

Fav outcomes $= 5$

Prob $= 5/2^{20}$

Favourable Subsets

① $S - \{7\}$

② $S - \{1, 6\}$

③ $S - \{2, 5\}$

④ $S \to \{3, 4\}$

⑤ $S \to \{1, 2, 4\}$

Nice - Subsets sum is 203

Results

AUB = A+ B = A or B denotes occurence of at least A or B. For 2 events A & B : **(See fig.1)**

(i) $P(A\cup B) = P(A) + P(B) - P(A\cap B) =$

(ii) Opposite of "atleast A or B" is **NIETHER A NOR B**

i.e. $\overline{A+B} = 1\text{-}(A \text{ or } B) = \overline{A} \cap \overline{B}$

Note that $P(A+B) + P(\overline{A} \cap \overline{B}) = 1.$

$\overline{A+B} = 1 - (A \text{ or } B)$

(iii) If A & B are mutually exclusive then $P(A\cup B) = P(A) + P(B).$

$P(\overline{A\cup B}) = 1 - P(A\cup B)$

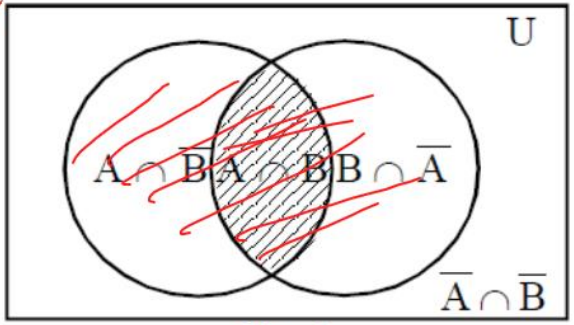


Fig . 1

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Results

For any three events A,B and C we have **(See Fig. 2)**

(i) $P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

(ii) P (at least two of A,B,C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 2P(A \cap B \cap C)$

(iii) P(exactly two of A,B,C occur) = $P(B \cap C) + P(C \cap A) + P(A \cap B) - 3P(A \cap B \cap C)$

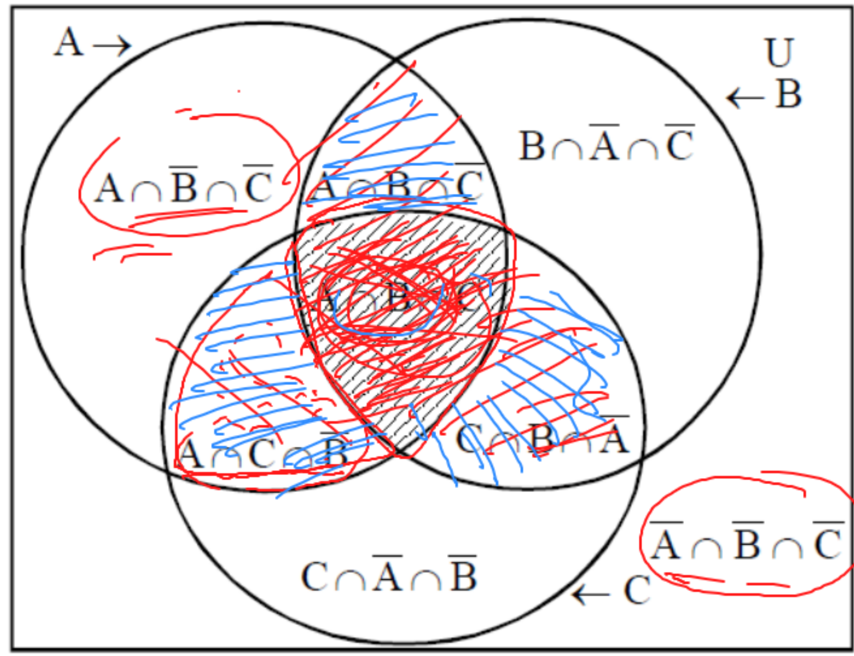


Fig. 2

(iv) P(exactly one of A,B,C occurs) = $P(A) + P(B) + P(C) - 2P(B \cap C) - 2P(C \cap A) - 2P(A \cap B) + 3P(A \cap B \cap C)$

Problems

For three events A, B and C, if P(exactly one of A or B occurs) = P(exactly one of B or C occurs) = P(exactly one of C or A occurs) = $\frac{1}{4}$ and P(all the three events occur simultaneously) = $\frac{1}{16}$, then the probability that atleast one of the events occurs, is **(2017 Main)**

(a) $\frac{7}{32}$ (b) $\frac{7}{16}$ (c) $\frac{7}{64}$ (d) $\frac{3}{16}$

$P(A\cup B) - P(A\cap B)$

$P(A)+P(B)-P(A\cap B)-P(A\cap B)$

$P(A)+P(B)-2P(A\cap B)$

① $P(A)+P(B)-2P(A\cap B)=\frac{1}{4}$

$P(A\cap B\cap C)=\frac{1}{16}$

② $P(B)+P(C)-2P(B\cap C)=1/4$

③ $P(C)+P(A)-2P(C\cap A)=1/4$

$P(A\cup B\cup C)= P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)+P(A\cap B\cap C)$

$\frac{3}{8}+\frac{1}{16} = \frac{6+1}{16} = \frac{7}{16}$

①+②+③

$2[P(A)+P(B)+P(C)-P(A\cap B)-P(B\cap C)-P(C\cap A)] = 3/4$

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Results

De Morgan's Law : – If A & B are two subsets of a universal set U , then

(a) $(A \cup B)^c = A^c \cap B^c$ & **(b)** $(A \cap B)^c = A^c \cup B^c$

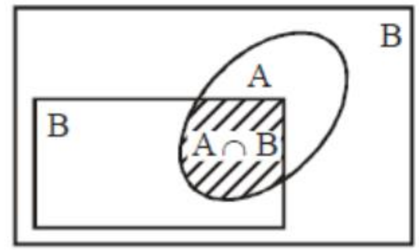
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ & $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$(A \cup B)^c = A^c \cap B^c.$

$(A \cap B)^c = A^c \cup B^c.$

Conditional Probability

Let A and B be any two events associated with a random experiment. The probability of occurrence of event A when the event B has already occurred is called the conditional probability of A when B is given and is denoted as P(A/B). The conditional probability P(A/B) is meaningful only when $P(B) \neq 0$, i.e., when B is not an impossible event.



By definition,

$P(A/B) \rightarrow$ Prob of occurrence of A when B has already occurred

$P(A \cap B) = P(B) \cdot P\left(\frac{A}{B}\right) = P(A) \cdot P\left(\frac{B}{A}\right)$

$$P\left(\frac{A}{B}\right) = \text{Probability of occurrence of event A when the event B as already occurred}$$

$$= \frac{\text{Number of cases favourable to B which are also favourable to A}}{\text{Number of cases favourable to B}}$$

$$\therefore \quad P\left(\frac{A}{B}\right) = \frac{\text{Number of cases favourable to } A \cap B}{\text{Number of cases favourable to B}}$$

$\boxed{P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}}$ ** $\boxed{P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}}$

If A & B are any two events $P(A \cap B) = P(A).P(B/A) = P(B).P(A/B)$, Where P(B/A) means conditional probability of B given A & P(A/B) means conditional probability of A given B.

Concepts

Independent Events **:** Two events A & B are said to be independent if occurence or non occurence of one does not effect the probability of the occurence or non occurence of other.

If the occurence of one event affects the probability of the occurence of the other event then the events are said to be **Dependent**

For two independent events A and B : $\boxed{P(A \cap B) = P(A).\ P(B).}$

Note : Independent events are not in general mutually exclusive & vice versa.
Mutually exclusiveness can be used when the events are taken from the same experiment & independence can be used when the events are taken from different experiments .

*Binomial Probability

The probability of getting exactly r success in n independent trials is given by

$\boxed{P(r) = {}^nC_r\, p^r q^{n-r}}$ where : p = probability of success in a single trial

q = probability of failure in a single trial. note : $\boxed{p + q = 1}$

Two outcomes → Success
→ Failure.

_ _ _ _ _ _ _ _ n places

In 'n' trials : Prob of getting exactly 'r' success will be

$${}^nC_r \times (p \times p \times p \times p - - - r \text{ times})\,(q \times q \times q - - - n-r \text{ times})$$

$$P(r\,\text{success}) = {}^nC_r \times p^r \times q^{n-r}$$

$p(s) = p = 1/2$

$q = 1/2$

A coin is tossed 10 times

Find the prob of getting exactly 7 heads

$$P(7\,\text{Heads}) = {}^{10}C_7 \times \left(\tfrac{1}{2}\right)^7 \times \left(\tfrac{1}{2}\right)^{10-7}$$

$$= {}^{10}C_7 \times \frac{1}{2^{10}}$$

Problems

Assume that each born child is equally likely to be a boy or a girl. If two families have two children each, then the conditional probability that all children are girls given that at least two are girls; is **(2019 Main, 10 April I)**

(a) $\frac{1}{17}$ (b) $\frac{1}{12}$ (c) $\frac{1}{10}$ (d) $\frac{1}{11}$

$P(B) = 1/2$

$P(G) = 1/2$.

$$P\left(\frac{\overbrace{4G}^{A}}{\underbrace{\text{Atleast } 2G}_{B}}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\text{All 4 girls}}{\text{All 4G} + 3G1B + 2G2B} = \frac{{}^4C_4 \times \left(\frac{1}{2}\right)^4}{{}^4C_4 \times \left(\frac{1}{2}\right)^4 + {}^4C_3 \times \left(\frac{1}{2}\right)^3 \times \frac{1}{2} + {}^4C_2 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^2}$$

$$= \frac{1/16}{\frac{1}{16} + \frac{4}{16} + \frac{6}{16}} = \frac{1}{11}.$$

Problems

For an initial screening of an admission test, a candidate is given fifty problems to solve. If the probability that the candidate can solve any problem is $\frac{4}{5}$, then the probability that he is unable to solve less than two problem is **(2019 Main, 12 April II)**

(a) $\frac{201}{5}\left(\frac{1}{5}\right)^{49}$ (b) $\frac{316}{25}\left(\frac{4}{5}\right)^{48}$ (c) $\frac{54}{5}\left(\frac{4}{5}\right)^{49}$ (d) $\frac{164}{25}\left(\frac{1}{5}\right)^{48}$

$P(C) = \frac{4}{5}$

$P(W) = 1/5.$

$P(49C\ 1W) + P(50C\ 0W)$

$^{50}C_{49} \times \left(\frac{4}{5}\right)^{49} \times \left(\frac{1}{5}\right)^{1} + {}^{50}C_{50} \times \left(\frac{4}{5}\right)^{50}$

$\frac{50 \times 4^{49}}{5^{50}} + \frac{4^{50}}{5^{50}} = \frac{4^{49}[50+4]}{5^{50}} = \frac{4^{49} \times 54}{5^{50}} = \frac{54}{5}\left(\frac{4}{5}\right)^{49}.$

Problems

The minimum number of times one has to toss a fair coin so that the probability of observing atleast one head is atleast 90% is (2019 Main, 8 April II)

(a) 2 (b) 3 (c) 5 (d) 4

P(atleast one head) = 1 − P(0 heads)

$= 1 - \left(\frac{1}{2}\right)^n$

$1 - \left(\frac{1}{2}\right)^n \geq \frac{9}{10}$

$\left(\frac{1}{2}\right)^n \leq \frac{1}{10}$

$n_{min} = 4$

Problems

Two integers are selected at random from the set { 1, 2, ... , 11}. Given that the sum of selected numbers is even, the conditional probability that both the numbers are even is **(2019 Main, 11 Jan I)**

(a) $\frac{2}{5}$ (b) $\frac{1}{2}$ (c) $\frac{7}{10}$ (d) $\frac{3}{5}$

Odd No: 1, 3, 5, 7, 9, 11 $\quad {}^6C_2$

Even No: 2, 4, 6, 8, 10 $\quad {}^5C_2$

A → both no. even.

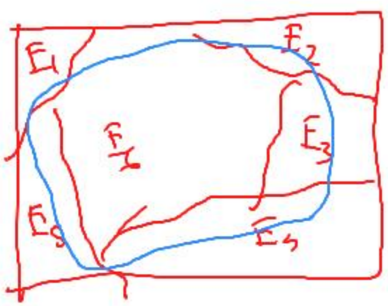
B → sum is even.

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{{}^5C_2 / {}^{11}C_2}{\left(\frac{{}^6C_2 + {}^5C_2}{{}^{11}C_2}\right)} = \frac{{}^5C_2}{{}^5C_2 + {}^6C_2} = \frac{10}{10+15} = \frac{10}{25} = \frac{2}{5}$$

TOTAL PROBABILITY THEOREM :

Let $E_1, E_2, \ldots\ldots, E_n$ be n mutually exclusive and exhaustive events, with non-zero probabilities, of a random experiment. If A be any arbitrary event of the sample space of the above random experiment with $P(A) > 0$, then

$$\boxed{P(A) = P(E_1)\,P\left(\frac{A}{E_1}\right) + P(E_2)\,P\left(\frac{A}{E_2}\right) + \ldots\ldots + P(E_n)\,P\left(\frac{A}{E_n}\right).}$$



$$P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) - \text{———} \; P(A \cap E_n)$$

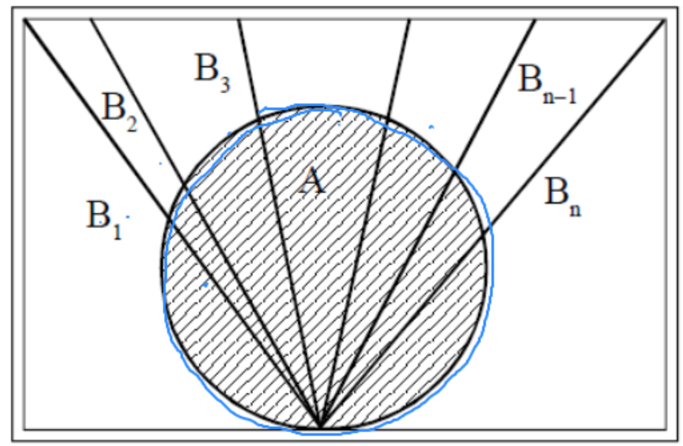
$$= P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right) \text{———}$$

Baye's Theorem

BAYE'S THEOREM :

If an event A can occur only with one of the n mutually exclusive and exhaustive events $B_1, B_2, \ldots . B_n$ & the probabilities $P(A/B_1)$, $P(A/B_2)$ .... $P(A/B_n)$ are known then,

$$P(B_1/A) = \frac{P(B_i).P(A/B_i)}{\sum_{i=1}^{n} P(B_i).P(A/B_i)}$$



$$P(A) = \sum P(B_i) \times P\left(\frac{A}{B_i}\right)$$

**

$$\boxed{P\left(\frac{B_1}{A}\right) = \frac{P(B_1 \cap A)}{P(A)} = \frac{P(B_1) \times P\left(\frac{A}{B_1}\right)}{\sum P(B_i) \times P\left(\frac{A}{B_i}\right)}}$$

Problems

A pot contain 5 red and 2 green balls. At random a ball is drawn from this pot. If a drawn ball is green then put a red ball in the pot and if a drawn ball is red, then put a green ball in the pot, while drawn ball is not replace in the pot. Now we draw another ball randomnly, the probability of second ball to be red is **(2019 Main, 9 Jan II)**

(a) $\frac{27}{49}$ (b) $\frac{26}{49}$ (c) $\frac{21}{49}$ (d) $\frac{32}{49}$

$P(G) = \frac{2}{7}$, $P(R) = \frac{5}{7}$

$2G, 5R \xrightarrow{R} 3G, 4R \xrightarrow{R_2} \frac{4}{7}$

$2G, 5R \xrightarrow{G} 1G, 6R \longrightarrow \frac{6}{7}$

$$P(R_2) = P(R) \times P\left(\frac{R_2}{R}\right) + P(G) \times P\left(\frac{R_2}{G}\right)$$

$$= \frac{5}{7} \times \frac{4}{7} + \frac{2}{7} \times \frac{6}{7}$$

$$= \frac{20}{49} + \frac{12}{49} = \frac{32}{49}.$$

MATHEMATICAL EXPECTATION

It is worthwhile indicating that if 'P' represents a person's chance of success in any venture and 'M 'the sum of money which he will receive in case of success, then the sum of money denoted by 'P·M' is called his expectation.

Ind vs Aus

70%   30%   M

$\frac{70}{100} \times M$ $\frac{30}{100} \times M$

Problems

$(3,6),(4,5),(5,4),(6,3)$

A person throws two fair dice. He wins ₹ 15 for throwing a doublet (same numbers on the two dice), wins ₹ 12 when the throw results in the sum of 9, and loses ₹ 6 for any other outcome on the throw. Then, the expected gain/loss (in ₹) of the person is

(2019 Main, 12 April II)

(a) $\frac{1}{2}$ gain (b) $\frac{1}{4}$ loss (c) $\frac{1}{2}$ loss (d) 2 gain

$P(\text{Doublet}) = \frac{6}{36} = \frac{1}{6}$

$P(\text{sum } 9) = \frac{4}{36} = \frac{1}{9}.$

$P(\text{rest}) = 1 - \frac{1}{6} - \frac{1}{9}$

$= 1 - \frac{6}{36} - \frac{4}{36} = \frac{26}{36}$

Expectation $= P_1 M_1 + P_2 M_2 + P_3 M_3$

$= \frac{1}{6} \times 15 + \frac{1}{9} \times 12 + \frac{26}{36} \times (-6)$

$= \frac{90 + 48 - 156}{36.} = -\frac{18}{36} = -\frac{1}{2}$

BINOMIAL PROBABILITY DISTRIBUTION :

Probability Distribution Table

Let an experiment has n independent trials and each of the trial has two possible out comes i.e. success or failure.
If getting number of successes in the experiment is a random variable then probability of getting exactly r-successes is -

$$P(x = r) = {}^{n}C_r\, p^r \cdot q^{n-r}$$

where p = probability of getting success
and q = probability of getting failure

x_i	p_i	$p_i x_i$	$p_i x_i^2$
0	${}^{n}C_0 p^0 q^n$	$0 \times {}^{n}C_0 p^0 q^n$	$0^2 \cdot {}^{n}C_0 p^0 q^n$
1	${}^{n}C_1 p^1 q^{n-1}$	$1 \times {}^{n}C_1 p^1 q^{n-1}$	$1^2 \cdot {}^{n}C_1 p^1 q^{n-1}$
2	${}^{n}C_2 p^2 q^{n-2}$	$\vdots$	$\vdots$
3	${}^{n}C_3 p^3 q^{n-3}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
r	${}^{n}C_r p^r q^{n-r}$	$r \times {}^{n}C_r p^r \cdot q^{n-r}$	$r^2 \cdot {}^{n}C_r p^r q^{n-r}$

Mean:-

Variance:

$\boxed{\text{Mean of BPD of a random variable}}$

$$\mu = \sum p_i x_i = \sum_{r=0}^{n} r \cdot {}^{n}C_r \cdot p^r \cdot q^{n-r} = \sum_{r=0}^{n} r\,\frac{n}{r} \cdot {}^{n-1}C_{r-1} \cdot p^r \cdot q^{n-r} = p \cdot n \sum_{r=1}^{n} {}^{n-1}C_{r-1} \cdot p^{r-1} q^{n-r}$$

$$= np\,[{}^{n-1}C_0 \cdot p^0\, q^{n-1} + {}^{n-1}C_1 \cdot p^1 q^{n-2} + \ldots\ldots + {}^{n-1}C_{n-1}\, p^{n-1}\, q^0]$$

$$= np\,(p+q)^{n-1} = np$$

Variance of BPD of a random variable :

$$\sigma^2 = \sum p_i x_i^2 - \mu^2$$

$$\sigma^2 = pn(1-p) = npq$$

Binomial Prob Distribution

$n \to$ total trials

$p \to$ prob of success in each trial

$q \to$ prob of failure in each trial.

$p+q=1$

μ = Mean = np

σ^2 = Variance = npq

S.D = $\sqrt{\text{Variance}} = \sqrt{npq}$

Standard deviation of BPD of a random variable :

Positive value of square root of variance is called standard deviation.

$$SD = +\sqrt{\sigma^2} = \sqrt{npq}$$

Problems

$^{16}C_2 = \frac{\cancel{16}^{8} \times 15}{\cancel{2}} = 120$

Let a random variable X have a binomial distribution with mean 8 and variance 4. If $P(X \le 2) = \frac{k}{2^{16}}$, then k is equal to (2019 Main, 12 April I)

(a) 17 (b) 121 (c) 1 (d) 137

$np = 8$

$npq = 4$.

$q = \frac{1}{2}$

$p = \frac{1}{2}$

$n = 16$

$P(X \le 2) = P(0\ \text{success}) + P(1\ \text{success}) + P(2\ \text{success})$

$= {}^{16}C_0 \times \left(\frac{1}{2}\right)^0 \times \left(\frac{1}{2}\right)^{16} + {}^{16}C_1 \times \left(\frac{1}{2}\right)^1 \times \left(\frac{1}{2}\right)^{15} + {}^{16}C_2 \times \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^{14}$

$= \frac{{}^{16}C_0 + {}^{16}C_1 + {}^{16}C_2}{\cancel{2^{16}}} = \frac{K}{\cancel{2^{16}}}$

$\Rightarrow 1 + 16 + 120 = K$

$K = 137$.
