Vector Algebra

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Definition

A VECTOR may be described as a quantity having both magnitude & direction. A vector is generally represented by a directed line segment, say $\vec{AB}$. A is called the **initial point** & B is called the **terminal point**. The magnitude of vector $\vec{AB}$ is expressed by $|\vec{AB}|$.

Types of Vectors

✓**Zero Vector** a vector of zero magnitude i.e.which has the same initial & terminal point, is called a **Zero Vector**. It is denoted by O.

✓**Unit Vector** a vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$.

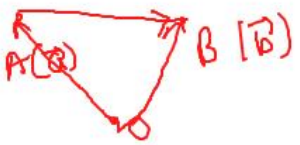
$\vec{a}$ $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$

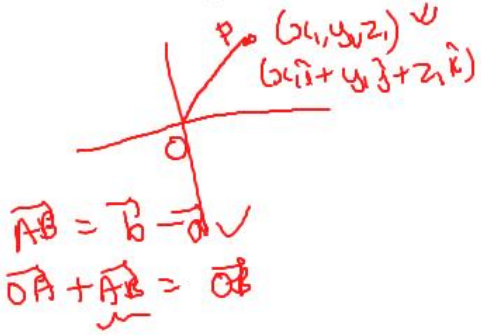
$\vec{a}=\hat{i}+2\hat{j}+3\hat{k}$

$|\hat{a}|=\frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$

✓**Position Vector** let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ & $\vec{b}$ & position vectors of two point A and B, then ,

$\overrightarrow{AB}=\vec{b}-\vec{a}$ = pv of B – pv of A.





Types of Vectors

COLLINEAR VECTORS two vectors are said to be collinear if their directed line segments are parallel disregards to their direction. Collinear vectors are also called **PARALLEL VECTORS**. If they have the same direction they are named as like vectors otherwise unlike vectors.

Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\boxed{\vec{a}=K\vec{b}}$, where $K \in R$

$\vec{a} = K\vec{b}$

COPLANAR VECTORS a given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"TWO VECTORS ARE ALWAYS COPLANAR"**.

Problems

Let $\vec{\alpha} = (\lambda - 2)\,\mathbf{a} + \mathbf{b}$ and $\vec{\beta} = (4\lambda - 2)\,\mathbf{a} + 3\mathbf{b}$ be two given vectors where vectors $\mathbf{a}$ and $\mathbf{b}$ are non-collinear. The value of λ for which vectors $\vec{\alpha}$ and $\vec{\beta}$ are collinear, is

(2019 Main, 10 Jan II)

(a) 4 (b) –3 (c) 3 (d) –4

If $\vec{\alpha}$ & $\vec{\beta}$ are collinear

$\vec{\beta} = k\vec{\alpha}$

$(4\lambda-2)\vec{a} + 3\vec{b} = k((\lambda-2)\vec{a} + \vec{b})$

$4\lambda - 2 = k(\lambda - 2)$

$3 = k$

$4\lambda - 2 = 3\lambda - 6$

$\lambda = -4$

VECTOR ADDITION :

If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.

$\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative) ☞ $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associativity)

$\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$ ☞ $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

MULTIPLICATION OF VECTOR BY SCALARS :

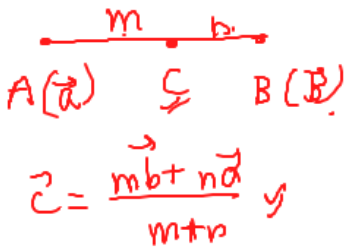
If $\vec{a}$ is a vector & m is a scalar, then $m\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called **SCALAR MULTIPLICATION**. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then:

$m(\vec{a})=(\vec{a})m=m\vec{a}$ $\qquad$ $m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

$(m+n)\vec{a}=m\vec{a}+n\vec{a}$ $\qquad$ $m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m + n}$. Note p.v. of mid point of AB $= \dfrac{\vec{a} + \vec{b}}{2}$.



Dot Product

* SCALAR PRODUCT OF TWO VECTORS :

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$0 \le \theta \le \pi$ — Angle b/w 2 vectors

- $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta(0 \le \theta \le \pi)$,

$\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

- $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2, \vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)
- $\vec{a}\,.\,(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)
- ** $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $(\vec{a} \neq 0 \quad \vec{b} \neq 0)$
- $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

$\vec{a}^2 = \vec{a}.\vec{a} = |\vec{a}||\vec{a}|\cos 0^\circ = |\vec{a}|^2$

- projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$. (Scalar)

Note: That vector component (vector) of $\vec{a}$ along $\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$ (vector component) and perpendicular to $\vec{b} = \vec{a} - \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$.

comp of $\vec{a}$ along $\vec{b}$ + comp of $\vec{a}$ ⊥ to $\vec{b}$ = $\vec{a}$

projection of $\vec{a}$ on $\vec{b}$ = $|\vec{a}|\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

Vector component of $\vec{a}$ on $\vec{b}$

$\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\,\hat{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\,\dfrac{\vec{b}}{|\vec{b}|}$

Dot Product

* SCALAR PRODUCT OF TWO VECTORS :

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$0 \le \theta \le \pi$ — Angle b/w 2 vectors

- $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta (0 \le \theta \le \pi)$,

$\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

- $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2, \vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)
- $\vec{a}.(\vec{b}+\vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)

** $\vec{a}^2 = \vec{a}.\vec{a} = |\vec{a}||\vec{a}|\cos 0^\circ = |\vec{a}|^2$

- **$\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$** $(\vec{a} \neq 0 \quad \vec{b} \neq 0)$
- $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$
- projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$. (scalar)

Note: That vector component of $\vec{a}$ along $\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$ (vector) — vector component

and perpendicular to $\vec{b} = \vec{a} - \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$.

comp of $\vec{a}$ along $\vec{b}$ + comp of $\vec{a}$ $\perp$ to $\vec{b}$ = $\vec{a}$

projection of $\vec{a}$ on $\vec{b} = |\vec{a}|\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

Vector component of $\vec{a}$ on $\vec{b}$

$\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\hat{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}\dfrac{\vec{b}}{|\vec{b}|}$

Note: That vector component of $\vec{a}$ along $\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular to $\vec{b} = \vec{a} - \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$.

☞ the angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $0 \le \phi \le \pi$

☞ if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$, $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$

Problems

If $\vec{a}$ and $\vec{b}_1$ are two unit vectors such that $\vec{a}+2\vec{b}$ and $5\vec{a}-4\vec{b}$, are perpendicular to each other, then angle between $\vec{a}$ and $\vec{b}$ is

(a) 45° (b) 60°

(c) $\cos^{-1}\left(\frac{1}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{7}\right)$

$(\vec{a}+2\vec{b})\cdot(5\vec{a}-4\vec{b})=0$

$5|\vec{a}|^2-4\vec{a}\cdot\vec{b}+10\vec{a}\cdot\vec{b}-8|\vec{b}|^2=0$

$5+6\vec{a}\cdot\vec{b}-8=0$

$6\vec{a}\cdot\vec{b}=3$

$\vec{a}\cdot\vec{b}=1/2$

$|\vec{a}||\vec{b}|\cos\theta=1/2$

$\cos\theta=1/2$

$\theta=60^\circ$

Note :

(i) Maximum value of $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|$ $(\cos\theta = 1)\ \theta = 0^\circ$

(ii) Minimum values of $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b} = -|\vec{a}|\,|\vec{b}|$ $(\cos\theta = -1)\ \theta = \pi$

(iii) Any vector $\vec{a}$ can be written as , $\vec{a} = (\vec{a} \cdot \hat{i})\,\hat{i} + (\vec{a} \cdot \hat{j})\,\hat{j} + (\vec{a} \cdot \hat{k})\,\hat{k}$.

Problems

Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$, $\mathbf{b} = b_1\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ and $\mathbf{c} = 5\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ be three vectors such that the projection vector of $\mathbf{b}$ on $\mathbf{a}$ is $\mathbf{a}$. If $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{c}$, then $|\mathbf{b}|$ is equal to

(2019 Main, 9 Jan II)

(a) 6 (b) 4 (c) $\sqrt{22}$ (d) $\sqrt{32}$

$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + 2}$

$= \sqrt{9 + 25 + 2}$

$= \sqrt{36} = 6$

$(\vec{a} + \vec{b}) \cdot \vec{c} = 0$

$\vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} = 0$

$5 + 1 + 2 + 5b_1 + b_2 + 2 = 0$

$5b_1 + b_2 = -10$ ✓

projection of $\vec{b}$ on $\vec{a}$ = $\left(\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2}\right)\vec{a} = \vec{a}$

$\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|^2} = 1 \Rightarrow \vec{b} \cdot \vec{a} = |\vec{a}|^2$

$b_1 + b_2 + 2 = 4$

$b_1 + b_2 = 2$

$4b_1 = -12$

$b_1 = -3$, $b_2 = 5$

VECTOR PRODUCT OF TWO VECTORS :

(i) ☞ If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta \hat{n}$,

where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\hat{n}$ forms a right handed screw system .

(ii) ☞ Lagranges Identity : for any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} \end{vmatrix}$

(iii) ☞ Formulation of vector product in terms of scalar product:

The vector product $\vec{a} \times \vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}| = \sqrt{\vec{a}^2\vec{b}^2 - (\vec{a} \cdot \vec{b})^2}$ (ii) $\vec{c} \cdot \vec{a} = 0$; $\vec{c} \cdot \vec{b} = 0$ and

(iii) $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed system

$(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

$|\vec{a}||\vec{b}|\sin\theta = 0 \Rightarrow \sin\theta = 0$

(iv) ☞ $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}$ & $\vec{b}$ are parallel (collinear) $(\vec{a}\neq 0, \vec{b}\neq 0)$ i.e. $\vec{a}=K\vec{b}$, where K is a scalar.

☞ $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ (not commutative)

☞ $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$ where m is a scalar.

$\hat{i}\times\hat{j}=\hat{k}$ $\quad \hat{k}\times\hat{i}=\hat{j}$

$\hat{j}\times\hat{k}=\hat{i}$

☞ $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (distributive)

☞ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ ☞ $\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$

(v) ☞ If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\ a_1&a_2&a_3\\ b_1&b_2&b_3\end{vmatrix}$

Problems

Let $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ be two vectors. If a vector perpendicular to both the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ has the magnitude 12, then one such vector is

(2019 Main 12 April II)

(a) $4(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ (b) $4(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

(c) $4(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ (d) $4(-2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

$\vec{a}+\vec{b} = 4\hat{i}+4\hat{j}$

$\vec{a}-\vec{b} = 2\hat{i}+4\hat{k}$

$\vec{c} = (\vec{a}+\vec{b}) \times (\vec{a}-\vec{b})$

$= (4\hat{i}+4\hat{j}) \times (2\hat{i}+4\hat{k})$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

$= \hat{i}(16) + \hat{j}(-16) + \hat{k}(-8)$

$\vec{c} = 16\hat{i} - 16\hat{j} - 8\hat{k}$

$\hat{c} = \frac{8(2\hat{i}-2\hat{j}-\hat{k})}{8\times 3} = \frac{2\hat{i}-2\hat{j}-\hat{k}}{3}$

$\pm 12\hat{c} = \pm 12\frac{4}{3}(2\hat{i}-2\hat{j}-\hat{k})$

$= \pm 4(2\hat{i}-2\hat{j}-\hat{k}) = 4(2\hat{i}-2\hat{j}-\hat{k})$

$(\vec{a}\times\vec{b})\cdot\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

- The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as :

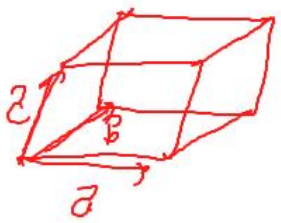
$\vec{a}\times\vec{b}.\vec{c} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a}\times\vec{b}$ & $\vec{c}$.

It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product .

- Scalar triple product geometrically represents the volume of the parallelopiped whose three couterminous edges are represented by $\vec{a}, \vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\ \vec{b}\ \vec{c}]$
- In a scalar triple product the position of dot & cross can be interchanged i.e.

$\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$ OR $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

- $\vec{a}.(\vec{b}\times\vec{c}) = -\vec{a}.(\vec{c}\times\vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$



$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

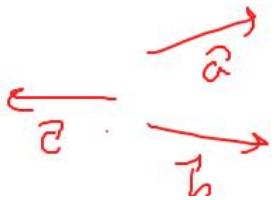
- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ & $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

In general , if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\,\vec{m}\,\vec{n}]$; where $\vec{\ell}$, $\vec{m}$ & $\vec{n}$ are non coplanar vectors .

- If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\,\vec{b}\,\vec{c}] = 0$.

If $\hat{a}, \hat{b}, \hat{c}$ are coplanar.



Problems

Let $\alpha \in R$ and the three vectors

$\mathbf{a} = \alpha\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \alpha\hat{\mathbf{k}}$

and $\mathbf{c} = \alpha\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Then, the set

$S = \{\alpha : \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar$\}$ (2019 Main, 12 April II)

(a) is singleton

(b) is empty

(c) contains exactly two positive numbers

(d) contains exactly two numbers only one of which is positive

$$\begin{vmatrix} \alpha & 1 & 3 \\ 2 & 1 & -\alpha \\ \alpha & -2 & 3 \end{vmatrix} = 0$$

Find α _ _

Problems

The sum of the distinct real values of μ, for which the vectors, $\mu\hat{i}+\hat{j}+\hat{k}, \hat{i}+\mu\hat{j}+\hat{k}, \hat{i}+\hat{j}+\mu\hat{k}$ are coplanar, is (2019 Main, 12 Jan I)

(a) 2 (b) 0 (c) 1 (d) -1

$$\begin{vmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{vmatrix} = 0$$

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Problems

Let $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + (\lambda^2 - 1)\hat{\mathbf{k}}$ be coplanar vectors. Then, the non-zero vector $\mathbf{a} \times \mathbf{c}$ is (2019 Main, 11 Jan I)

(a) $-10\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ (b) $-10\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$

(c) $-14\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$ (d) $-14\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$ $\begin{vmatrix} 1 & 2 & 4 \\ 1 & \lambda & 4 \\ 2 & 4 & \lambda^2-1 \end{vmatrix} = 0$ Find λ & obtain $\vec{c}$

$\vec{a} \times \vec{c}$ ✓

VECTOR TRIPLE PRODUCT :

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}$

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the expression $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector & is called a vector triple product.

GEOMETRICAL INTERPRETATION OF $\vec{a} \times (\vec{b} \times \vec{c})$

Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ & $(\vec{b} \times \vec{c})$. Now $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ & $(\vec{b} \times \vec{c})$ but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane $\vec{b}$ & $\vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector lies in the plane of $\vec{b}$ & $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of $\vec{b}$ & $\vec{c}$ i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars.

- $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$
- $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Problems

$a = 3$

Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c}$ be a vector such that $|\mathbf{c} - \mathbf{a}| = 3$, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$ and the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ is 30°. Then, $\mathbf{a} \cdot \mathbf{c}$ is equal to

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(a) $\frac{25}{8}$ (b) 2 (c) 5 (d) $\frac{1}{8}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = 2\hat{i} + \hat{j}(-2) + \hat{k}(1)$$

$$|\vec{a} \times \vec{b}| = 3$$

$$|(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}|\,|\vec{c}| \sin 30^\circ = 3$$

$$|\vec{a} \times \vec{b}|\,|\vec{c}| = 6$$

$$3 \times |\vec{c}| = 6 \Rightarrow \boxed{|\vec{c}| = 2}$$

$$|\vec{c} - \vec{a}| = 3$$

$$|\vec{c} - \vec{a}|^2 = 9$$

$$(\vec{c} - \vec{a}) \cdot (\vec{c} - \vec{a}) = 9$$

$$\vec{c}^2 + \vec{a}^2 - 2\,\vec{a} \cdot \vec{c} = 9 \Rightarrow \vec{a} \cdot \vec{c} = \frac{\vec{c}^2 + \vec{a}^2 - 9}{2} = \frac{\vec{c}^2}{2} = \frac{2^2}{2} = 2$$

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