3D Geometry

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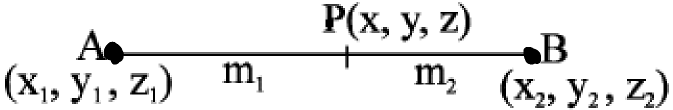
Definition

A. General :

(1) Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)

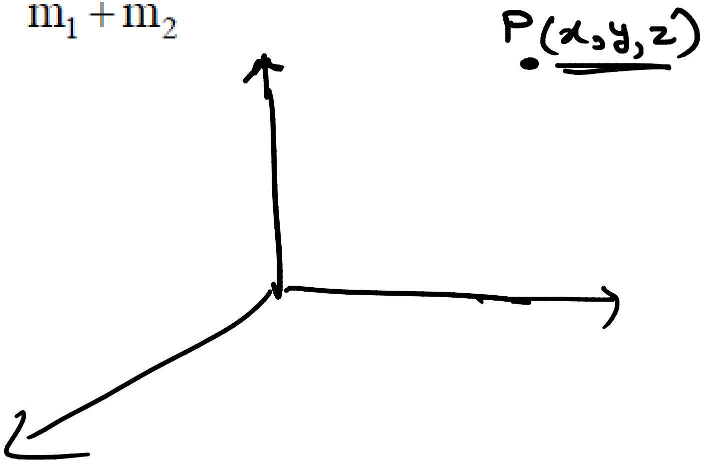
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



(2) **Section Fomula**

$$x = \frac{m_2 x_1 + m_1 x_2}{m_1 + m_2} \quad ; \quad y = \frac{m_2 y_1 + m_1 y_2}{m_1 + m_2} \quad ; \quad z = \frac{m_2 z_1 + m_1 z_2}{m_1 + m_2}$$

(For external division take –ve sign)



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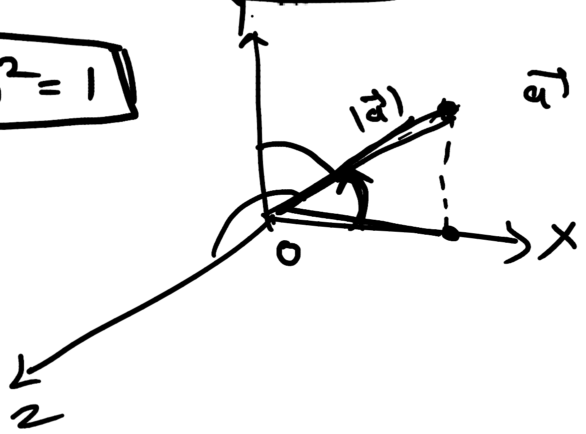
DIRECTION COSINES : (cosine of Direction angle)

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, the angles which this vector makes with the +ve directions OX, OY & OZ are called **Direction Angles** & their cosines are called the **Direction Cosines**.

$\cos\alpha = \dfrac{a_1}{|\vec{a}|}$, $\cos\beta = \dfrac{a_2}{|\vec{a}|}$, $\cos\gamma = \dfrac{a_3}{|\vec{a}|}$.

Note that, $\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$

$$\boxed{l^2 + m^2 + n^2 = 1}$$



Direction Cosine and direction ratio's of a line

~~(3)~~ Direction cosine of a line has the same meaning as d.c's of a vector.

~~(a)~~ Any three numbers a, b, c proportional to the direction cosines are called the direction ratios i.e.

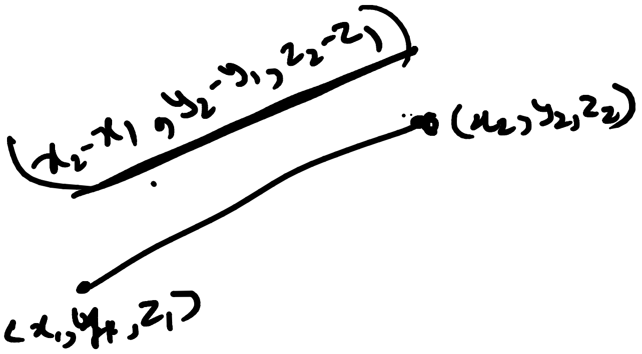
l, m, n

(a, b, c)

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

same sign either +ve or –ve should be taken through out.

note that d.r's of a line joining x_1, y_1, z_1 and x_2, y_2, z_2 are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$



(b) If θ is the angle between the two lines whose d.c's are l_1, m_1, n_1 and l_2, m_2, n_2

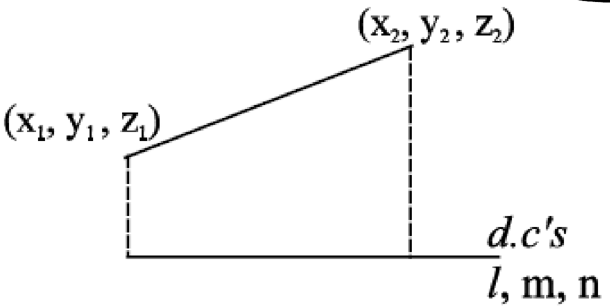
$$\boxed{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2}$$

$$\boxed{\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}}$$

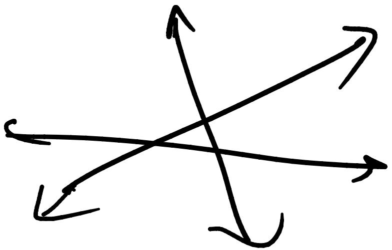
(i) hence if lines are perpendicular then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

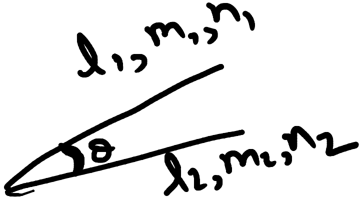
(ii) if lines are parallel then $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

note that if three lines are coplanar then $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$



(4) Projection of the join of two points on a line with d.c's l, m, n are
$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$





Problems

9)

The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$, is

(2014 Main)

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

$\frac{1}{2} = \cos\theta$

$\boxed{\theta = \pi/3}$

$l^2 + m^2 + n^2 = 1$

$2l^2 = 1$

$l = \left(\pm\frac{1}{\sqrt{2}}\right)$

$-l = (m+n)$

$l^2 = m^2 + n^2 + 2mn$

$\boxed{mn = 0}$

$\boxed{l = \frac{1}{\sqrt{2}}, n = 0, m = -\frac{1}{\sqrt{2}}}$, $l = \frac{1}{\sqrt{2}}, m = 0, n = -\frac{1}{\sqrt{2}}$

EQUATION OF A PLANE :

The equation $(\vec{r} - \vec{r}_0).\vec{n} = 0$ represents a plane containing the point with p.v. $\vec{r}_0$ where $\vec{n}$ is a vector normal to the plane. $\vec{r}.\vec{n} = d$ is the general equation of a plane.

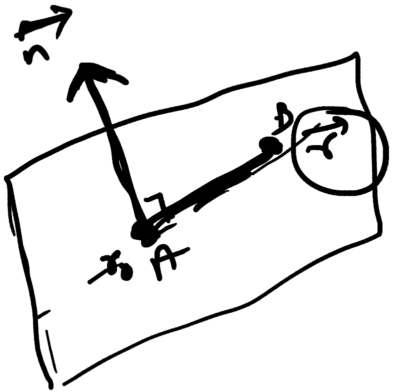
$AB = (\vec{r} - \vec{r_0})$

Vector eqn

$$\boxed{(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0}$$

$$\vec{r} \cdot \vec{n} = \vec{r_0} \cdot \vec{n}$$

$$\boxed{\vec{r} \cdot \vec{n} = d}$$



$$(a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) = d$$

$\vec{n}$

$$\boxed{a_1 x + b_1 y + c_1 z = d}$$

Cartesian eqn of Plane

$\boxed{2x + 3y + 5z = 1}$

B PLANE

(i) General equation of degree one in x, y, z i.e $\boxed{ax + by + cz + d = 0}$ represents a plane.

(ii) Equation of a plane passing through (x_1, y_1, z_1) is

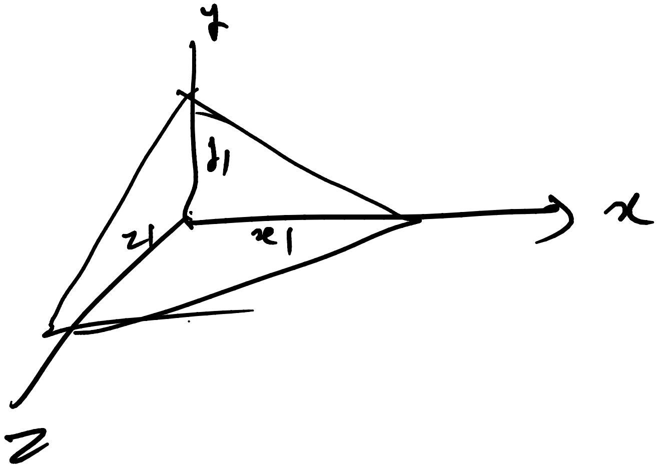
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where a, b, c are the direction ratios of the normal to the plane.

a, b, c $\longrightarrow$ $\vec{n}$

(iii) Equation of a plane if its intercepts on the co-ordinate axes are x_1, y_1, z_1 is

$$\boxed{\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1.}$$

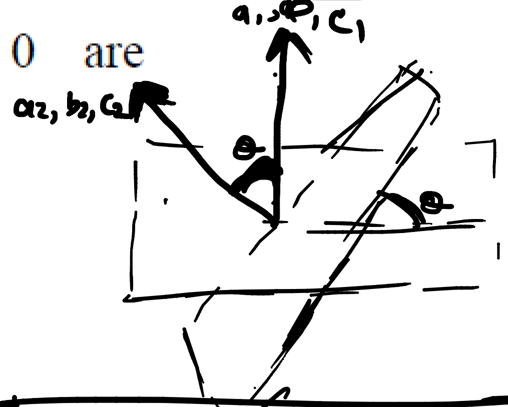


(v) **Parallel and perpendicular planes** – Two planes $a_1 x + b_1 y + c_1 z + d_1 = 0$ and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are

perpendicular if $\boxed{a_1 a_2 + b_1 b_2 + c_1 c_2 = 0}$

parallel if $\boxed{\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}}$ and $\neq \dfrac{d_1}{d_2}$

coincident if $\boxed{\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = \dfrac{d_1}{d_2}}$

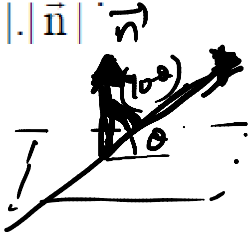


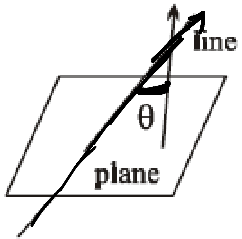
$$\boxed{\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\,\sqrt{a_2^2+b_2^2+c_2^2}}}$$

(vi) Angle between a plane and a line is the compliment of the angle between the normal to the plane and the line. If $\boxed{\begin{array}{l} \text{Line} \ : \vec{r} = \vec{a} + \lambda(\vec{b}) \\ \text{Plane} : \vec{r}.\vec{n} = d \end{array}}$ then $\cos(90-\theta) = \sin\theta = \dfrac{\vec{b}.\vec{n}}{|\vec{b}|.|\vec{n}|}$

where θ is the angle between the line and normal to the plane.

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$

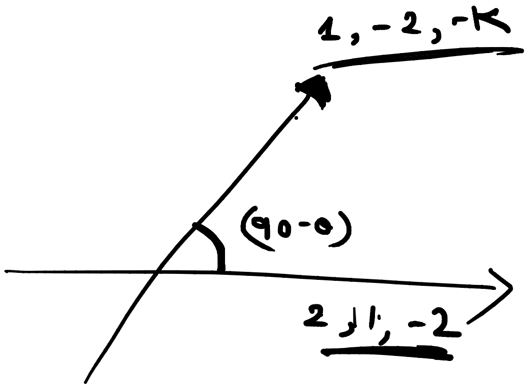




Problems

9) If an angle between the line, $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and the plane, $x - 2y - kz = 3$ is $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$, then value of k is (2019 Main, 12 Jan II)

(a) $\sqrt{\frac{5}{3}}$ (b) $\sqrt{\frac{3}{5}}$ (c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$



$\sin\theta = \frac{\cancel{2+1+4}\ \cancel{(1\times 2)} + (-2\times -k)}{\sqrt{5+k^2}\cdot\sqrt{9}}$

$\sin\theta = \frac{2k}{3\sqrt{k^2+5}}$

$\theta = \sin^{-1}\left(\frac{2k}{3\sqrt{k^2+5}}\right)$

$2k$, $3\sqrt{k^2+5}$, θ, $\sqrt{5(k^2+9)}$

$9(k^2+5) - 4k^2$
$5k^2 + 45$
$5(k^2+9)$

$\cos\theta = \frac{\sqrt{5(k^2+9)}}{3\sqrt{k^2+5}} = \frac{2\sqrt{2}}{3}$

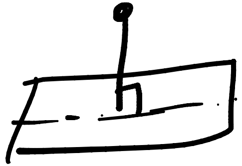
$5(k^2+9) = 8(k^2+5)$

$5 = 3k^2$

$k = \sqrt{5/3}$

(vii) Length of the perpendicular from a point (x_1, y_1, z_1) to a plane $ax + by + cz + d = 0$ is

$$p = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$



(viii) Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

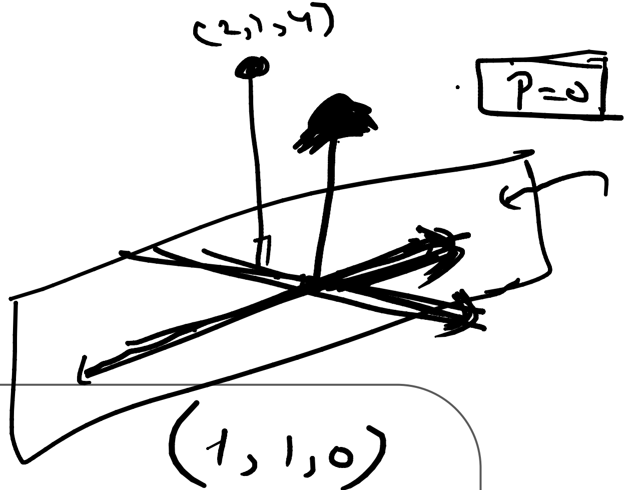
$$\left| \frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Problems

9) ★★

The length of the perpendicular drawn from the point (2, 1, 4) to the plane containing the lines $\mathbf{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\mathbf{r} = (\hat{i} + \hat{j}) + \mu(-\hat{i} + \hat{j} - 2\hat{k})$ is (2019 Main, 12 April II)

(a) 3 (b) $\frac{1}{3}$ ~~(c) $\sqrt{3}$~~ (d) $\frac{1}{\sqrt{3}}$



$(1, 2, -1)$ $(-1, 1, -2)$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix}$$

$\hat{i}(-3) - \hat{j}(-3) + \hat{k}(3)$

$-3, 3, 3$

$\boxed{-1, 1, 1} \rightarrow$ (D^r of $\vec{n}$)

$\left|\frac{3}{\sqrt{3}}\right|$

$-1(x-1) + 1(y-1) + 1(z-0) = 0$

$\boxed{-x + y + z = 0}$
