Problem Solving on Matrices

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INVERSE OF A MATRIX

A square matrix A said to be invertible if and only if it is non-singular (i.e. $|A| \neq 0$) and there exists a matrix B such that, $AB = I = BA$.

B is called the **inverse** (reciprocal) of A and is denoted by $\mathbf{A^{-1}}$. Thus

$$A^{-1} = B \Leftrightarrow AB = I = BA$$

$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

$$A \,.\, (\text{adj } A) = |A| \; I_n$$

$$A^{-1} \; A \; (\text{adj } A) = A^{-1} I_n \; |A|$$

$$I_n \; (\text{adj } A) = A^{-1} \; |A| \; I_n$$

$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 3 & 1 & 2 \end{bmatrix}, \quad A^{-1} = ?$$

$$\text{adj} A = [C]^T \qquad C = \begin{bmatrix} (-2) & 8 & (-1) \\ -1 & -7 & 5 \\ 4 & -5 & 2 \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} -2 & -1 & 4 \\ 8 & -7 & -5 \\ -1 & 5 & 2 \end{bmatrix},$$

$$|A| = (-2) - 2(-8) + 3(-1)$$

$$= 11$$

$$A^{-1} = \frac{1}{11} \begin{bmatrix} -2 & -1 & 4 \\ 8 & -7 & -5 \\ -1 & 5 & 2 \end{bmatrix}$$

PROPERTIES OF INVERSE:

$(A) \longrightarrow A^{-1}$

Theorem–1 : Every invertible matrix possesses a unique inverse.

Theorem-2 : If A is an invertible square matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

$(A^T)^{-1} = (A^{-1})^T$

Theorem-3 : If A is a non-singular matrix, then prove that

$|A| \neq 0$

$$|A^{-1}| = |A|^{-1} \qquad \text{i.e.} \qquad \boxed{|A^{-1}| = \frac{1}{|A|}}$$

Theorem-4 : If A & B are invertible matrices of the same order , then $\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$.

$A = [\;]$

$|A^{-1}| = \frac{1}{|A|}$ $\quad A^{-1}$ $(\;)$

1. If A is a symmetric matrix and B is a skew-symmetric matrix such that $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, then AB is equal to

(2019 Main, 12 April I)

(a) $\begin{bmatrix} -4 & -2 \\ -1 & 4 \end{bmatrix}$ ~~(b)~~ $\begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -2 \\ 1 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & 2 \\ 1 & 4 \end{bmatrix}$

$$\boxed{\begin{aligned} A &= A^T \\ B &= -B^T \end{aligned}}$$

$$A+B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} \quad -(1)$$

$$(A+B)^T = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix} \quad -(2)$$

(1+2)

$$2A = \begin{bmatrix} 4 & 8 \\ 8 & -2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}$$

(1) - (2)

$$2B = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$$

Q)

Let $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}, (\alpha \in R)$ such that $A^{32} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then, a value of α is

4M

(2019 Main, 8 April I)

(a) $\frac{\pi}{32}$ (b) 0 ~~(c)~~ $\frac{\pi}{64}$ (d) $\frac{\pi}{16}$

$32\alpha = \pi/2$

$\boxed{\alpha = \frac{\pi}{64}}$

1 Min

$$A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & +\cos 2\alpha \end{bmatrix}$$

$$A^{32} = \begin{bmatrix} \cos 32\alpha & -\sin 32\alpha \\ \sin 32\alpha & +\cos 32\alpha \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

JEE Advanced 2016

lower Triangular

Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}$ and $Q = [q_{ij}]$ be two 3×3 matrices such that $Q - P^5 = I_3$. Then, $\dfrac{q_{21} + q_{31}}{q_{32}}$ is equal to **(2019 Main, 12 Jan I)**

(a) ~~10~~ (b) 135 (c) 9 (d) 15

$\dfrac{q_{21} + q_{31}}{q_{32}}$

$q = ?$

$q = I_3 + P^5$

$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} +$

$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 27 & 6 & 1 \end{bmatrix}$

$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 27 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & 0 \\ 9 & 1 & 0 \\ 54 & 9 & 1 \end{bmatrix}$

$P^5 + \begin{bmatrix} 2 & 0 & 0 \\ 15 & 2 & 0 \\ 135 & 15 & 2 \end{bmatrix}$

$54 + 36 = 90$

$90 + 45 = 135$

$\dfrac{\cancel{150}}{\cancel{15}} = 10$

9) Let $A = \begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix}$. If $AA^T = I_3$, then $|p|$ is (2019 Main, 11 Jan I)

(a) $\frac{1}{\sqrt{5}}$ ~~(b)~~ $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{6}}$

$$\boxed{2q^2 = r^2}$$

$$3q^2 = 1 - 3q^2$$

$$\begin{bmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{bmatrix} \begin{bmatrix} 0 & p & p \\ 2q & q & -q \\ r & -r & r \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4q^2+r^2 & 2q^2-r^2 & -2q^2+r^2 \\ 2q^2-r^2 & p^2+q^2+r^2 & p^2-q^2-r^2 \\ -2q^2+r^2 & p^2-q^2-r^2 & p^2+q^2+r^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{p^2+q^2+r^2 = 1}, \quad p^2 = q^2 + r^2$$

$$p^2 = \boxed{1-3q^2} \qquad \boxed{p^2 = 3q^2}$$

$$q^2 = \frac{1}{6}$$

$$p^2 = 3 \times \frac{1}{6} = \frac{1}{2}$$

$$\pm \frac{1}{\sqrt{2}}$$