A value of $\theta \in (0, \pi/3)$, for which

$$\begin{vmatrix} 1+\cos^2\theta & \sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & 1+\sin^2\theta & 4\cos 6\theta \\ \cos^2\theta & \sin^2\theta & 1+4\cos 6\theta \end{vmatrix} = 0, \text{ is}$$

(2019 Main, 12 April II)

(a) $\frac{\pi}{9}$ (b) $\frac{\pi}{18}$ (c) $\frac{7\pi}{24}$ (d) $\frac{7\pi}{36}$

Special Determinants

Factorisation in respect the following determinants are very useful and should be remembered.

(i) $$\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = (x-y)(y-z)(z-x)$$

(ii) $$\begin{vmatrix} x^3 & x & 1 \\ y^3 & y & 1 \\ z^3 & z & 1 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

(iii) $$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Cramers Rule

Simultaneous linear equations

- Consistent system (at least one solution)
  - Exactly one solution or Unique solution
    - Trivial solution: All variable zero is the only solution
      - $x = 0, y = 0, z = 0$
    - Non trivial solution: At least one non zero variable satisfies the system
      - $x = 2, y = 0, z = 0$
      - $x = -1, y = 2, z = 3$
  - Infinite solutions
- Inconsistent System (no solution)

$$2x + 5y = 3$$
$$3x + 2y = 9$$

$$x + 2y + 3z = 2$$

CRAMER'S RULE : [Simultaneous Equations Involving Three Unknowns]

Let $a_1x + b_1y + c_1z = d_1$ ...(I) ;

$a_2x + b_2y + c_2z = d_2$ ...(II) ;

$a_3x + b_3y + c_3z = d_3$ ...(III)

$x = ?,\ y = ?,\ z = ?$

Then, $x = \dfrac{D_1}{D}$, $Y = \dfrac{D_2}{D}$, $Z = \dfrac{D_3}{D}$.

Where $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; $D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$; $D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ & $D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

unique

$\boxed{x = \dfrac{D_1}{D},\ y = \dfrac{D_2}{D},\ z = \dfrac{D_3}{D}}$

Trivial

$\boxed{D \neq 0}$, $\boxed{D_1, D_2, D_3 = 0}$ $\boxed{x = 0, y = 0, z = 0}$

Non-

$\boxed{D \neq 0}$ $D_1 \neq 0,\ D_2 = D_3 = 0$ $\boxed{x \neq 0,\ y = 0,\ z = 0}$

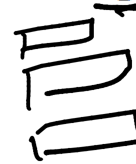
$D = 0$ $D_1 \neq 0, 1$ No soln.

Cramers Rule

~~(a)~~ If $D \neq 0$ and alteast one of $D_1, D_2, D_3 \neq 0$, then the given system of equations are consistent and have unique non trivial solution.

~~(b)~~ If $D \neq 0$ & $D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have trivial solution only.

~~(c)~~ If $D = D_1 = D_2 = D_3 = 0$, then the given system of equations are consistent and have infinite solutions.



$x + 2y + 3z = d_1$
$2x + 4y + 6z = d_2$
$3x + 6y + 9z = d_3$

* Exception

In case $\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\}$ represents these parallel planes then also

$D = D_1 = D_2 = D_3 = 0$ but the system is inconsistent. (No soln)

~~(d)~~ If $D = 0$ but atleast one of D_1, D_2, D_3 is not zero then the equations are inconsistent and have no solution.

D

$D \neq 0$ → Consistent, (unique soln)
- $D_1, D_2, D_3 = 0$ (Trivial)
- $D_1 \neq 0, D_2 = D_3$ Non-Trivial

$D = 0$
- $D_1, D_2, D_3 = 0$ (Infinite) consist
- $D_1 \neq 0, D_2 = 0, D_3$ No soln

MULTIPLICATION OF TWO DETERMINANTS

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 l_2 & a_1 m_1 + b_1 m_2 \\ a_2 l_1 + b_2 l_2 & a_2 m_1 + b_2 m_2 \end{vmatrix} = \begin{vmatrix} a_1 l_1 + b_1 m_1 & a_1 l_2 + b_1 m_2 \\ a_2 l_1 + b_2 m_1 & a_2 l_2 + b_2 m_2 \end{vmatrix}$$

Similarly two determinants of order three are multiplied.

Here we have multiplied row by column. We can also multiply row by row, column by row and column by column.

$(a_1 l_1 + b_1 l_2 + c_1 l_3)$

$$\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$$

$(5-6) = (2-3)$

$-1 = -1$

If a, b, c x, y, z ∈ R, then prove that $\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} = \begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix}$

$$a_1 x + \cancel{b_1} y = c_1 \quad \text{— (1)}$$

$$a_2 x + b_2 y = c_2 \quad \text{— (2)}$$

$$x + 2y = 1$$

$$2x + 3y = 2$$

$$x = \left(\frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1}\right) \qquad y = \left(\frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}\right)$$

$$\boxed{x} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \left(\frac{D_1}{D}\right) \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad \left(\frac{D_2}{D}\right)$$

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \underline{D_1} = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$\left.\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}\right\}$$ (eq^n of st. line)

$\underline{a_1b_2 - a_2b_1 \neq 0}$

$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

(1) $\boxed{\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}}$ $\rightarrow$ $\boxed{D \neq 0}$ $\rightarrow$ $\underline{\text{Unique}}$ sol^n

(2) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$ ($\underline{D=0}$, $\underline{D_1 \neq 0, D_2 \neq 0}$) $\rightarrow$ $\underline{\text{No}}$ $\underline{sol^n}$

(3) $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ $\underline{(D=0, D_1=0, D_2=0)}$ $\rightarrow$ $\underline{\text{Infinite } sol^n}$

$$\left(x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}\right)$$

Cramer's soln

D

- $\boxed{D \neq 0}$ (Unique)
  - Trivial: If $\boxed{D_1 = D_2 = D_3 = 0}$
  - Non-trivial: $\boxed{\text{If atleast one of } (D_1, D_2, D_3) \neq 0}$
- $\boxed{D = 0}$
  - atleast one $\boxed{D_1, D_2, D_3 \neq 0}$ — NO soln
  - $\boxed{D_1 = D_2 = D_3 = 0}$ — Infinite soln

$$
\begin{aligned}
7x - 7y + 5z &= 3 \\
3x + y + 5z &= 7 \\
2x + 3y + 5z &= 5
\end{aligned}
$$