Trigonometric Ratios

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IMPORTANT T′ RATIOS:

$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin\frac{(2n+1)\pi}{2} = (-1)^n$ & $\cos\frac{(2n+1)\pi}{2} = 0$ where $n \in I$

(c) $\sin 15^\circ$ or $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75^\circ$ or $\cos\frac{5\pi}{12}$;

$\sin\theta = \cos(90-\theta)$

$\cos 15^\circ$ or $\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75^\circ$ or $\sin\frac{5\pi}{12}$;

$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75^\circ$; $\tan 75^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15^\circ$

(d) $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$; $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$; $\tan\frac{\pi}{8} = \sqrt{2}-1$; $\tan\frac{3\pi}{8} = \sqrt{2}+1$

$\sin 22.5^\circ$ $\cos 22.5^\circ$ $\tan 22.5^\circ = \sqrt{2}-1$

(e) $\sin\frac{\pi}{10}$ or $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$ & $\cos 36^\circ$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

$\sin 18^\circ = \frac{\sqrt{5}-1}{4}$, $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$

Trigonometric Functions Of Allied Angles :

If θ is any angle, then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **Allied Angles**.

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

(b) $\sin(90°-\theta) = \cos\theta$; $\cos(90°-\theta) = \sin\theta$

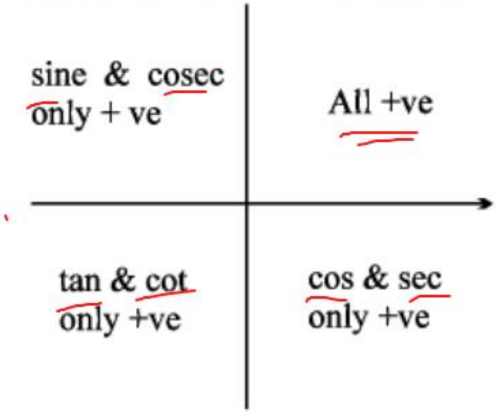
(c) $\sin(90°+\theta) = \cos\theta$; $\cos(90°+\theta) = -\sin\theta$

(d) $\sin(180°-\theta) = \sin\theta$; $\cos(180°-\theta) = -\cos\theta$

(e) $\sin(180°+\theta) = -\sin\theta$; $\cos(180°+\theta) = -\cos\theta$

(f) $\sin(270°-\theta) = -\cos\theta$; $\cos(270°-\theta) = -\sin\theta$

(g) $\sin(270°+\theta) = -\cos\theta$; $\cos(270°+\theta) = \sin\theta$



$\sin 210° = \sin(180°+30°) = -\sin 30° = -1/2$

Rule No: 1

$f_1(n\pi \pm \theta) = f_1(\theta)$

$f_1(n\frac{\pi}{2} \pm \theta) = f_2(\theta)$ n is odd.

complimentary fn's

$\sin\left(\frac{3\pi}{2}-\alpha\right) = -\cos\alpha$ $\sin(5\pi-\alpha) = \sin\alpha.$

$\sin(3\pi+\alpha) = -\sin\alpha.$

$\sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha$

Problems

The expression

$$3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$$
$$-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$$

is equal to

(a) 0   (b) 1

(c) 3   (d) $\sin 4\alpha + \cos 6\alpha$

$$E \equiv 3[\cos^4\alpha + \sin^4\alpha] - 2[\cos^6\alpha + \sin^6\alpha]$$
$$= 3[1 - 2\sin^2\alpha\cos^2\alpha] - 2[1 - 3\sin^2\alpha\cos^2\alpha]$$
$$= 3 - 2 - 6\cancel{\sin^2\alpha}\cos^2\alpha + 6\cancel{\sin^2\alpha}\cos^2\alpha$$
$$= 1$$

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$$\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta.$$
$$= 1 - 2\sin^2\theta\cos^2\theta$$

$$\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$$

Problems

If $\tan\theta = -\frac{4}{3}$, then $\sin\theta$ is

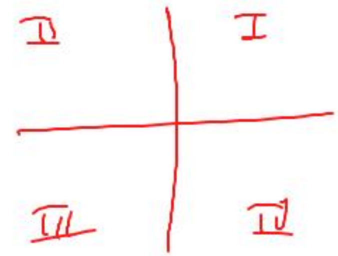
(a) $-\frac{4}{5}$ but not $\frac{4}{5}$

(b) $-\frac{4}{5}$ or $\frac{4}{5}$

(c) $\frac{4}{5}$ but not $-\frac{4}{5}$

(d) None of the above

$|\sin\theta| = \frac{P}{h} = \frac{4}{5}$



$\tan\theta$ is -ve.

$\theta \in \text{II}$ or $\theta \in \text{IV}$

$\sin\theta = \frac{4}{5}$ $\quad$ $\sin\theta = -\frac{4}{5}$

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TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF TWO ANGLES :

(a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B) \cdot \sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B) \cdot \cos(A-B)$

(e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

(f) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

Problems

If $\alpha + \beta = \dfrac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan\alpha$ equals

(a) $2(\tan\beta + \tan\gamma)$ (b) $\tan\beta + \tan\gamma$

(c) $\tan\beta + 2\tan\gamma$ (d) $2\tan\beta + \tan\gamma$

$\alpha + \beta = \frac{\pi}{2}$

$\alpha = \frac{\pi}{2} - \beta$

$\tan\alpha = \tan\left(\frac{\pi}{2} - \beta\right)$

$\tan\alpha = \cot\beta$

$\boxed{\tan\alpha \tan\beta = 1}$

$\beta + \gamma = \alpha$

$\gamma = \alpha - \beta$

$\tan\gamma = \tan(\alpha - \beta)$

$\tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1+1} \Rightarrow 2\tan\gamma = \tan\alpha - \tan\beta$

$\tan\alpha = \tan\beta + 2\tan\gamma.$

Factorisation Of The Sum Or Difference Of Two sines Or cosines :

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(b) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(d) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

Transformation Of Products Into Sum Or Difference Of sines & cosines :

(a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(b) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(d) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Problems

$1+\cos 2\theta = 2\cos^2\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$2\cos A\cos B = \cos(A+B)+\cos(A-B)$

The value of $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$ is

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(a) $\frac{3}{2}(1+\cos 20°)$ (b) $\frac{3}{4}+\cos 20°$

(c) $3/2$ (d) $3/4$

$\frac{1}{2}\left[2\cos^2 10° - 2\cos 10°\cos 50° + 2\cos^2 50°\right]$

$= \frac{1}{2}\left[1+\cos 20° - [\cos 60° + \cos 40°] + 1+\cos 100°\right]$

$= \frac{1}{2}\left[1+\cos 20° - \frac{1}{2} - \cos 40° + 1 + \cos 100°\right]$

$= \frac{1}{2}\left[\frac{3}{2} + \underbrace{\cos 20° + \cos 100°} - \cos 40°\right]$

$= \frac{1}{2}\left[\frac{3}{2} + \cancel{2}\cancel{\cos 60°}\cdot\cos 40° - \cos 40°\right]$

$= \frac{1}{2}\left[\frac{3}{2} + \cancel{\cos 40°} - \cancel{\cos 40°}\right]$

$= \frac{3}{4}.$

Multiple Angles And Half Angles :

(a) $\boxed{\sin 2A = 2\sin A\cos A}$; $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$;

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$.

$2\cos^2 A = 1 + \cos 2A$, $2\sin^2 A = 1 - \cos 2A$; $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

$2\cos^2\frac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\frac{\theta}{2} = 1 - \cos\theta$.

(c) $\boxed{\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}}$; $\tan\theta = \dfrac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$

(d) $\boxed{\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}}$, $\boxed{\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Problems

$\tan(\alpha+\beta) = \frac{4}{3}$, $\tan(\alpha-\beta) = 5/12$.

If $\cos(\alpha+\beta) = \frac{3}{5}$, $\sin(\alpha-\beta) = \frac{5}{13}$ and $0 < \alpha, \beta < \frac{\pi}{4}$, then $\tan(2\alpha)$ is equal to **(2019 Main, 8 April I)**

(a) $\frac{63}{52}$ ~~(b)~~ $\frac{63}{16}$ (c) $\frac{21}{16}$ (d) $\frac{33}{52}$

$$\tan(2\alpha) = \tan(\underbrace{\alpha+\beta} + \underbrace{\alpha-\beta})$$

$$= \frac{\tan(\alpha+\beta) + \tan(\alpha-\beta)}{1 - \tan(\alpha+\beta)\cdot\tan(\alpha-\beta)}$$

$$= \frac{\frac{4}{3} + 5/12}{1 - \frac{4}{3} \times \frac{5}{12}} = \frac{\frac{16+5}{12}}{\frac{36-20}{36\,3}} = \frac{21 \times 3}{16} = 63/16$$

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Note :

$$\sin A \;\sin(60° - A)\;\sin(60° + A) = \frac{1}{4}\sin 3A$$

$$\cos A \;\cos(60° - A)\;\cos(60° + A) = \frac{1}{4}\cos 3A$$

$$\tan A \;\tan(60° - A)\;\tan(60° + A) = \tan 3A$$

The value of sin 20° sin 40° sin 60° sin 80° is -

(A) $\frac{3}{8}$ (B) $\frac{1}{8}$ (C) $\frac{3}{16}$ (D) None of these

$$\frac{\sqrt{3}}{2}\left[\sin 20° \sin(60-20°) \cdot \sin(60+20°)\right]$$

$$\frac{\sqrt{3}}{2} \times \frac{1}{4}\sin(3\times 20°) = \frac{1}{4}\times\frac{\sqrt{3}}{2}\times\frac{\sqrt{3}}{2} = \frac{3}{16}$$

Problems

The value of $\sin 10^{\circ} \sin 30^{\circ} \sin 50^{\circ} \sin 70^{\circ}$ is

(2019 Main, 9 April II)

(a) $\frac{1}{36}$ (b) $\frac{1}{32}$ (c) $\frac{1}{16}$ (d) $\frac{1}{18}$

$\frac{1}{2}\left(\sin 10^{\circ} \cdot \sin 50^{\circ} \cdot \sin 70^{\circ}\right)$

$\frac{1}{2}\left[\sin 10^{\circ} \cdot \sin(60^{\circ}-10^{\circ}) \cdot \sin(60^{\circ}+10^{\circ})\right]$

$\frac{1}{2} \times \frac{1}{4} \sin(10^{\circ} \times 3)$

$= \frac{1}{8} \sin 30^{\circ}$

$= \frac{1}{8} \times \frac{1}{2} = 1/16$

Continued product of cosine Series :

$$\cos \underline{A} \cos \underline{2A} \cos \underline{4A} \cos \underline{8A} \ldots \cos 2^{n-1} A = \frac{1}{2^n \sin A} \sin (2^n A)$$

Angles are in G.P with c.r = 2

~~Cos 10°~~

$$\frac{2^5 \sin 10^\circ}{2^5 \sin 10^\circ} \cos 10^\circ \cdot \cos 20^\circ \cos 40^\circ \cos 80^\circ \cos 160^\circ$$

$$= \frac{2^4 \sin 20^\circ \cdot \cos 20^\circ \cos 40^\circ \cos 80^\circ \cos 160^\circ}{2^5 \sin 10^\circ}$$

$$= \frac{2^3 \sin 40^\circ \cos 40^\circ \cos 80^\circ \cos 160^\circ}{2^5 \sin 10^\circ} = \frac{\sin 320^\circ}{2^5 \sin 10^\circ}$$

THREE ANGLES:

(a) $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

NOTE IF : **(i)** $A+B+C=\pi$ then $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(ii) $A+B+C=\dfrac{\pi}{2}$ then $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$

(b) If $A+B+C=\pi$ then : **(i)** $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$

(ii) $\sin A+\sin B+\sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS:

(a) Min. value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $\theta \in R$

(b) Max. and Min. value of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2+b^2}$ and $-\sqrt{a^2+b^2}$

$a^2\tan^2\theta + b^2\cot^2\theta$ Min value $= 2ab$ A.M / G.M inequalities

$a\cos\theta + b\sin\theta$ Min value $= -\sqrt{a^2+b^2}$ Max $= +\sqrt{a^2+b^2}$

Sum of sines or cosines of n angles,

Angles are in A.P.

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots + \sin\left(\alpha + \overline{n-1}\,\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \frac{n-1}{2}\beta\right)$$

First term $=\alpha$, c.d $=\beta$

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos\left(\alpha + \overline{n-1}\,\beta\right) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \frac{n-1}{2}\beta\right)$$

Problems

Find the sum of series $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$.

$\alpha = \pi/11 \quad \beta = \frac{2\pi}{11}, \quad n = 5$

$$\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \cos\left(\alpha + (n-1)\frac{\beta}{2}\right) = \frac{\sin 5\times\frac{\pi}{11}}{\sin\frac{\pi}{11}} \cdot \cos\left(\frac{\pi}{11} + \frac{4\pi}{11}\right)$$

$$= \frac{2\sin\frac{5\pi}{11}\cdot\cos\frac{5\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{\sin\frac{10\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{\sin\left(\pi - \frac{\pi}{11}\right)}{2\sin\frac{\pi}{11}}$$

$$= \frac{\cancel{\sin\pi/11}}{2\cancel{\sin\pi/11}} = \frac{1}{2}.$$

An equation (or inequality) involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation.

$$\cos x = \frac{1}{2};\ \sin^2 x - 4\cos x = 1.$$

There are millions and billions of solutions which satisfy say tan x = – 1 but our main object is to write down those infinite solution in one line. Since all trigonometric function are periodic and therefore solution of all trigonometrical equation can be generalised with the help of periodicity of trigonometrical function.

$\sin x = 1/2$ → 2 principal soln

→ infinite general soln.

Two types of solution

Principal solution

$[0, 360°)$

General solution

~~all~~

Principal Solution

$\theta = 30^\circ, 150^\circ$

The solutions of a trigonometric equation lying in the interval $[0, 2\pi)$. For example, $\sin\theta = \frac{1}{2}$, then the two values of θ between 0 and 2π are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Thus, $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are the principal solutions of equation $\sin\theta = \frac{1}{2}$.

General Solution

The solution
consisting of all possible solutions of a trigonometric equation is called its general solution.

Trigonometric Equations

1. If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.
2. If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.
3. If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.
4. If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
5. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
6. $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

$\theta = n\pi \pm \alpha.$

[Note : α is called the principal angle]

Trigonometric Equations

1. If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n \alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$.
2. If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.
3. If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$.
4. If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
5. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
6. $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$. [Note: α is called the principal angle]

$\sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \sin\frac{\pi}{6}$

$\theta = 30^\circ, 150^\circ$

$\theta = n\pi + (-1)^n \frac{\pi}{6}$ $n \in I$

TYPES OF TRIGONOMETRIC EQUATIONS :

$a\sin x + b\cos x = c$ Divide both sides by $\sqrt{a^2+b^2}$

$1\sin x + 1\cos x = \sqrt{2}$

$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = 1$

$\cos\frac{\pi}{4}\sin x + \sin\frac{\pi}{4}\cos x = 1$

$\sin(x+\frac{\pi}{4}) = 1$

(a) Solutions of equations by factorising. Consider the equation ;
$(2\sin x - \cos x)(1+\cos x) = \sin^2 x$; $\cot x - \cos x = 1 - \cot x \cos x$

(b) Solutions of equations reducible to quadratic equations. Consider the equation :
$3\cos^2 x - 10\cos x + 3 = 0$ and $2\sin^2 x + \sqrt{3}\sin x + 1 = 0$

(c) Solving equations by introducing an Auxilliary argument. Consider the equation :
$\sin x + \cos x = \sqrt{2}$; $\sqrt{3}\cos x + \sin x = 2$; $\sec x - 1 = (\sqrt{2} - 1)\tan x$

$\sin(x+\frac{\pi}{4}) = \sin\frac{\pi}{2}$

$x + \frac{\pi}{4} = n\pi + (-1)^n\frac{\pi}{2}$

$$\boxed{x = n\pi + (-1)^n\frac{\pi}{2} - \frac{\pi}{4}}$$

a) $(2\sin x - \cos x)(1+\cos x) = \sin^2 x$

$\Rightarrow (2\sin x - \cos x)(1+\cos x) = (1-\cos x)(1+\cos x)$

$(1+\cos x)(2\sin x - \cancel{\cos x} - 1 + \cancel{\cos x}) = 0$

$(1+\cos x)(2\sin x - 1) = 0$

$\cos x = -1$ $\sin x = 1/2$

$\sin x = 1/2$

$x = n\pi + (-1)^n\frac{\pi}{6}$ — (1)

$\cos x = -1 = \cos\pi$

$x = 2m\pi \pm \pi$. — (2)

(1) $\cup$ (2)