(70%-80%)

3 LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER :

The most general form of a linear differential equations of first order is $\boxed{\frac{dy}{dx} + Py = Q}$, where P & Q are functions of x .

To solve such an equation multiply both sides by $e^{\int P dx}$.

$$\boxed{\frac{dy}{dx} + P(x)\cdot y = g(x)}$$

Solve $\frac{dy}{dx} + by = e^{nx}$.

1. Find Integrating factor

$$\boxed{I.F = e^{\int p(x)dx}}$$

2. Sol^n

$$\boxed{y \times I.F = \int g(x)\cdot(I.F)\, dx}$$

9)

$$\frac{dy}{dx} + \frac{y}{(1+x^2)} = \frac{e^{\tan^{-1} x}}{1+x^2}$$

$x = 0$ $y = 1$

$$y(0) = 1$$

$$\boxed{\frac{dy}{dx} + P(x)y = g(x)}$$

1. $I.F = e^{\int \frac{1}{1+x^2} dx}$

$$= \left(e^{\tan^{-1} x}\right)$$

2. $\underline{y \times e^{\tan^{-1} x}} = \int \left(\frac{e^{\tan^{-1} x}}{1+x^2}\right) \times \left(e^{\tan^{-1} x}\right) dx$

$$= \int \frac{e^{2\tan^{-1} x}}{1+x^2} dx$$

$\tan^{-1} x = t$

$\frac{1}{1+x^2} dx = dt$

$$= \int e^{2t} dt$$

$$\boxed{y \cdot e^{\tan^{-1} x} = \frac{e^{2\tan^{-1} x}}{2} + C}$$

Eq^n Reducible to Linear

* **Bernoulli's Equation:** (Type of Linear Differential eq^n)

$$\frac{dy}{dx} + Py = Qy^n \qquad (1)$$

where P and Q are function of x alone or constants. Divide each term of (1) by y^n. We get

$$\boxed{\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q} \qquad (2)$$

$$\boxed{\frac{1}{y^{n-1}} = v}$$

$$y^{1-n} = v$$

Let $\frac{1}{y^{n-1}} = v$, so that $\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$

$$(1-n)\, y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$$

Substituting in (2), we get

$$\frac{dv}{dx} + (1-n)v.P = Q(1-n) \qquad (3)$$

(3) is a linear differential equation

$$\left(\frac{1}{1-n}\right)\frac{dv}{dx} + v\,P(x) = q(x)$$

v

$$\boxed{\frac{dv}{dx} + P(x)(1-n)\,v = q(x)(1-n)}$$

$$I.F = e^{\int P(x)(1-n)\,dx}$$

$$v \times I.F = \int q(x)(1-n)\,I.F\,dx$$

Solve the differential equation $x\frac{dy}{dx}+y=x^3y^6$.

$$\frac{x\,dy}{y^6\,dx}+\frac{1}{y^5}=x^3$$

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{y^5}\left(\frac{1}{x}\right)=x^2$$

$$\boxed{\frac{1}{y^5}=t}$$

$$-\frac{5}{y^6}\frac{dy}{dx}=\frac{dt}{dx}$$

$$\left(\frac{-1}{5}\right)\frac{dt}{dx}+\frac{t}{x}=x^2$$

$$\boxed{\frac{dt}{dx}+\left(-\frac{5}{x}\right)t=-5x^2}$$

$$I\cdot F=e^{\int -5/x\,dx}$$

$$=e^{-5\ln x}$$

$$I\cdot F=e^{\ln(x^{-5})}=x^{-5}$$

$$t\times\frac{1}{x^5}=\int\frac{-5x^2}{x^5}\,dx$$

$$\frac{t}{x^5}=-5\int\frac{1}{x^3}\,dx+$$

ORTHOGONAL TRAJECTORY

$\perp$, (90°)

Any curve, which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family. For example, each straight line passing through the origin, i.e. $y = kx$ is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$

Procedure for finding the orthogonal trajectory

(i) Let $f(x, y, c) = 0$ be the equation of the given family of curves, where c is an arbitrary parameter.

(ii) Differentiate $f = 0$, w.r.t. x and eliminate c i.e. form a differential equation.

(iii) Substitute $-\frac{dx}{dy}$ for $\left(\frac{dy}{dx}\right)$ in the above differential equation. This will give the differential equation of the orthogonal trajectory.

(i) $\boxed{x^2+y^2=a^2}$

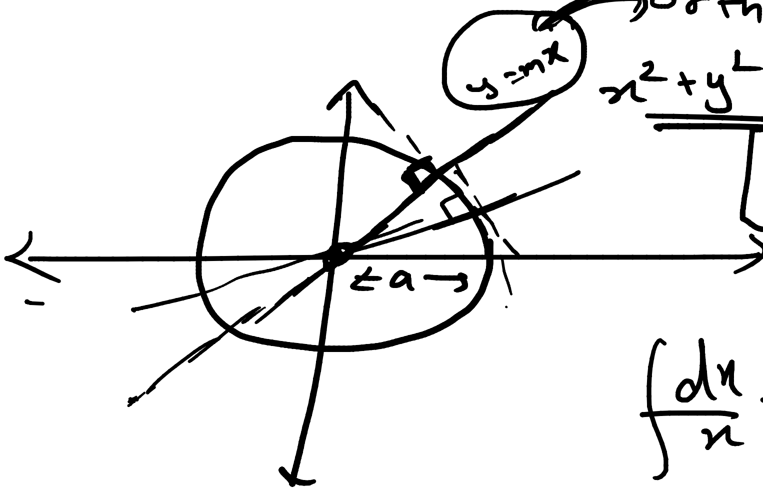
(ii) $2x + 2y\frac{dy}{dx} = 0$

$\left(\frac{dy}{dx}\right) = -\frac{x}{y}$

(iii) $+\frac{dx}{dy} = +\frac{x}{y}$

orthogonal trajectory

$x^2+y^2=a^2$



$\int \frac{dx}{x} = \int \frac{dy}{y}$

$\ln x = \ln y + c$

$\boxed{\ln y = \ln x - \ln c}$

$\boxed{y = mx}$

Find the orthogonal trajectories of the family of ellipses $\boxed{x^2 + 2y^2 - y = c.}$

(i) $x^2 + 2y^2 - y = c$

$$2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(4y-1) = -2x$$

(ii) $$\frac{dy}{dx} = \frac{-2x}{4y-1}$$

(iii) $$+\frac{dx}{dy} = \frac{+2x}{(4y-1)}$$

$$\int\frac{dx}{2x} = \int\frac{dy}{4y-1}$$

$4y-1 = t$

$4\,dy = dt$

$$\boxed{\frac{1}{2}\ln x = \frac{1}{4}\ln|4y-1| + c}$$

$$\boxed{\ln x = \ln|4y-1|^{1/2} + \ln c}$$

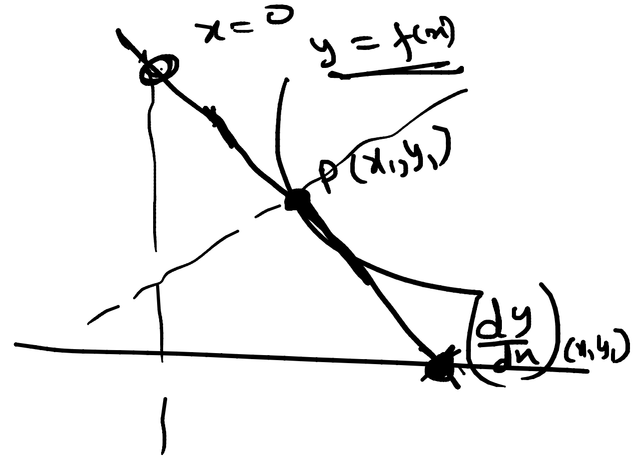
$$\ln x = \ln(c\cdot(4y-1)^{1/2})$$

$$\boxed{x = c(4y-1)^{1/2}}$$

$$\boxed{x^2 = c(4y-1)}$$

Geometrical applications :

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at point P is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$



(i) The equation of the tangent at P is $y - y_1 = \frac{dy}{dx}(x - x_1)$

x-intercept of the tangent $= x_1 - y_1\left(\frac{dx}{dy}\right)$

y-intercept of the tangent $= y_1 - x_1\left(\frac{dy}{dx}\right)$

(ii) The equation of normal at P is $y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$

$$m_n = -\frac{1}{m_t}$$

Exact Differential Equation

$\int x^2 dx$ $d\left(\frac{x^3}{3}\right)$

d $d(xy)$

$\int (y\,dx + x\,dy) = xy$

$\int (\quad)$

$\frac{d}{}(x+y) = \int (dx + dy)$

$d(x+y)$

For this the following results must be memorized.

(i) $d(x+y) = dx + dy$

(ii) $d(xy) = y\,dx + x\,dy$

(iii) $d\left(\frac{x}{y}\right) = \boxed{\frac{y\,dx - x\,dy}{y^2}}$

(iv) $d\left(\frac{y}{x}\right) = \boxed{\frac{x\,dy - y\,dx}{x^2}}$

(v) $d(\log xy) = \frac{y\,dx + x\,dy}{xy}$

(vi) $d\left(\log\frac{y}{x}\right) = \frac{(x\,dy - y\,dx)}{xy}$

(vii) $d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right) = \frac{x\,dy - y\,dx}{x^2 - y^2}$

(viii) $\boxed{d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}}$

(ix) $\frac{d[f(x,y)]^{1-n}}{1-n} = \frac{f'(x,y)}{(f(x,y))^n}$

$\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{x\,dy - y\,dx}{x^2}\right)$

(x) $d\left(\sqrt{x^2+y^2}\right) = \frac{x\,dx + y\,dx}{\sqrt{x^2+y^2}}$