Complex Numbers

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Concepts

Complex numbers are expressions of the form $a + ib$ where $a, b \in R$ & $i = \sqrt{-1}$. It is denoted by z i.e. $\boxed{z = a + ib}$. 'a' is called as real part of z (Re z) and 'b' is called as imaginary part of z (Im z).

$z = a + ib$

purely real	purely imaginary	imaginary
if $b = 0$	if $a = 0$	if $b \neq 0$
$z = a$	$z = ib$	

Note that :

(i) The symbol i combines itself and with real number as per the rule of algebra together with $i^2 = -1$; $i^3 = -i$; $i^4 = 1$, $i^{2005} = i$; $i^{2006} = -1$. Infact $\boxed{i^{4n} = 1\ ,\ n \in I}$]

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Problems

The value of $i^{57} + 1/i^{125}$ is :-

(A) 0 (B) $-2i$ (C) $2i$ (D) 2

$i^{57} = i^{56} \times i^{1}$

$= i$

$i^{125} = i^{124} \times i = i$

$i + \frac{1}{i} = i + \frac{i^3}{i^4} = i + \frac{-i}{1} = i - i = 0$

Concepts

ALGEBRAIC OPERATIONS :

The algebraic operations on complex numbers are similiar to those on real numbers treating *i* as a polynomial. Inequalities in complex numbers are not defined. There is no validity if we say that complex number is positive or negative.

e.g. $z > 0$, $4 + 2i < 2 + 4i$ are meaningless .

$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$

$z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2)$

$z_1 \times z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + i^2 y_1y_2 + x_1y_2 i + x_2y_1 i$

$= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$

$\frac{z_1}{z_2} =$

EQUALITY IN COMPLEX NUMBER :

Two complex numbers $z_1 = a_1 + ib_1$ & $z_2 = a_2 + ib_2$ are equal if and only if their real & imaginary parts coincide.

$a_1 = a_2$ & $b_1 = b_2$

$\frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2} = \frac{\quad}{x_2^2 + y_2^2}$

Problems

Given that $x, y \in R$, solve : (a) $(x + 2y) + i\,(2x - 3y) = 5 - 4i$

$$\left.\begin{aligned} x + 2y &= 5 \\ 2x - 3y &= -4 \end{aligned}\right] \text{ solve.}$$

Three Important terms : Conjugate / Modulus / Argument

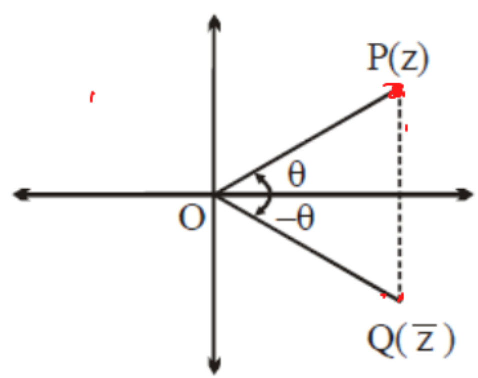
CONJUGATE COMPLEX :

If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part & is denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

$z = 3i + 4$

$\bar{z} = -3i + 4$

$z\bar{z} = (a+ib)(a-ib) = a^2 - i^2b^2 = a^2 + b^2$



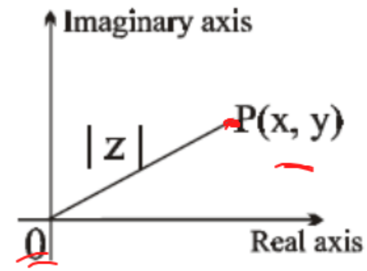
Note that :

(i) $z + \bar{z} = 2\,\text{Re}(z)$ **(ii)** $z - \bar{z} = 2i\,\text{Im}(z)$ **(iii)** $z\bar{z} = a^2 + b^2$ which is real

(iv) If z lies in the 1st quadrant then $\bar{z}$ lies in the 4th quadrant and $-\bar{z}$ lies in the 2nd quadrant.

Three Important terms : Conjugate / Modulus / Argument

Modulus of Complex Number :

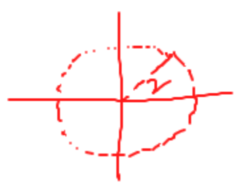


Modulus of complex number is a distance of the point on the argand plane representing the complex number z from the origin.

If P denotes a complex number $z = x + iy$

then $OP = |z| = \sqrt{x^2 + y^2}$ note that $|z| \geq 0$. ; $|i| = 1$; i.e. $|\sqrt{-1}|$

All complex number having the same modulus lie on a circle with centre as origin and radius $r = |z|$.

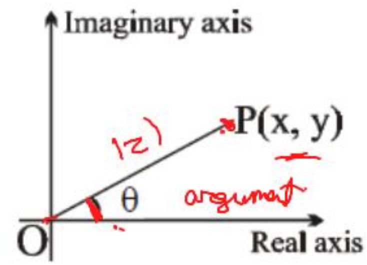


$|z| = 2$

Three Important terms : Conjugate / Modulus / Argument

Argument of Complex Number :

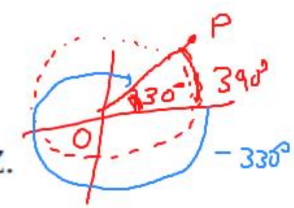
Angle (θ) made by the line segment joining the point on the complex plane representing the complex number z to the origin from the positive real axis is called argument of complex number z which is denoted as $\arg(z) = \theta$.



$z = x + iy$

(x, y)

General Argument :

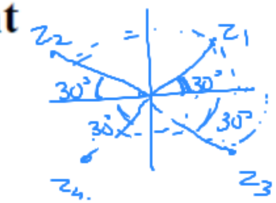


If OP makes an angle θ with real axis then θ is called one of the argument of z. General values of argument of z are given by

$2n\pi + \theta,\ n \in I$. Note that any two argument differ by 2π.

all the arguments

Three Important terms : Conjugate / Modulus / Argument



amp$(z_1) = 30°$
amp$(z_2) = 150°$
amp$(z_3) = -30°$
amp$(z_4) = -150°$

Amplitude (Principal value of argument) :

**

The unique value of θ such that $\boxed{-\pi < \theta \le \pi}$ is called principal value of argument. Unless otherwise stated, amp z refers to the principal value of argument.

$z = x + iy$ $\quad$ $\tan\alpha = \left|\frac{y}{x}\right| \Rightarrow \alpha = \tan^{-1}\frac{y}{x}$

Quadrant	Sign of x & y	Arg(z)
I	x, y>0	$\tan^{-1}\frac{y}{x}$
II	x<0, y>0	$\pi - \tan^{-1}\left\|\frac{y}{x}\right\|$
III	x, y < 0	$-\pi + \tan^{-1}\left\|\frac{y}{x}\right\|$
IV	x>0, y<0	$-\tan^{-1}\left\|\frac{y}{x}\right\|$

2nd Quadrant: $Arg(z) = \pi - \alpha$

1st Quadrant: $Arg(z) = \alpha$

3rd Quadrant: $Arg(z) = -\pi + \alpha$ $\quad -(\pi - \alpha)$

4th Quadrant: $Arg(z) = -\alpha$

IMPORTANT PROPERTIES OF CONJUGATE / MODULI / AMPLITUDE :

(a) $z + \bar{z} = 2\,\mathrm{Re}\,(z)$; $z - \bar{z} = 2\,i\,\mathrm{Im}\,(z)$; $\overline{(\bar{z})} = z$; $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$;

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$; $\overline{z_1 z_2} = \bar{z}_1 \,.\, \bar{z}_2$ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$

$z = a + ib$, $\bar{z} = a - ib$

$\bar{\bar{z}} = a + ib = z$

$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

$\overline{z_1 z_2} = \bar{z}_1 \cdot z_2$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

IMPORTANT PROPERTIES OF CONJUGATE / MODULI / AMPLITUDE :

(b) $|z| \geq 0$; $|z| \geq \text{Re}(z)$; $|z| \geq \text{Im}(z)$; $\boxed{|z| = |\bar{z}| = |-z|}$; $z\bar{z} = |z|^2$;

$|z_1 z_2| = |z_1| . |z_2|$; $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$, $|z^n| = |z|^n$;

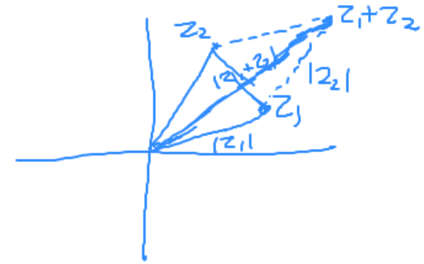
$|z \times z \times z \times z - -| = |z||z||z||z| - -$

$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\ [|z_1|^2 + |z_2|^2]$

$\boxed{||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|}$ **[TRIANGLE INEQUALITY]**

$z = a + ib$, $|z| = \sqrt{a^2+b^2}$

$z \cdot \bar{z} = (a+ib)(a-ib)$
$= a^2 + b^2 = |z|^2$



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IMPORTANT PROPERTIES OF CONJUGATE / MODULI / AMPLITUDE :

(c) **(i)** $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi.$ $k \in I$

(ii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi$; $k \in I$

(iii) $\text{amp}(z^n) = n\,\text{amp}(z) + 2k\pi$.

where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

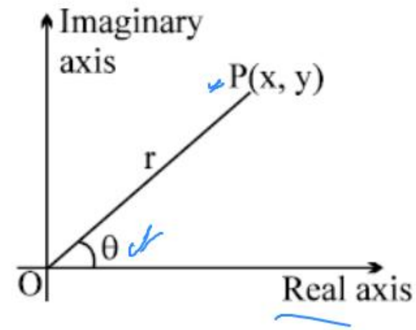
$\text{amp}(z \times z \times z \ldots) = \text{amp } z + \text{amp } z + \text{amp } z \ldots$

$= n(\text{amp } z) + 2k\pi$

REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(a) Cartesian Form (Geometric Representation) :

Every complex number $z = x + iy$ can be represented by a point on the cartesian plane known as complex plane (Argand diagram) by the ordered pair (x, y).



length OP is called modulus of the complex number denoted by $|z|$ & θ is called the argument or amplitude .

eg. $|z| = \sqrt{x^2 + y^2}$ &

$\theta = \tan^{-1}\frac{y}{x}$ (angle made by OP with positive x–axis)

Problems

Find the locus of :

(a) $|z-1|^2 + |z+1|^2 = 4$ (b) $Re(z^2) = 0$

a) $z = x + iy$

$|x+iy-1|^2 + |x+iy+1|^2 = 4$

$\left(\sqrt{(x-1)^2+y^2}\right)^2 + \left(\sqrt{(x+1)^2+y^2}\right)^2 = 4$

$x^2+1-\cancel{2x}+y^2 + x^2+1+\cancel{2x}+y^2 = 4$

$2x^2+2y^2 = 2$

$\boxed{x^2+y^2=1}$ Circle.

b) $z = x+iy$

$z^2 = (x+iy)^2 = x^2+i^2y^2+2xyi$

$= x^2-y^2+2xyi$

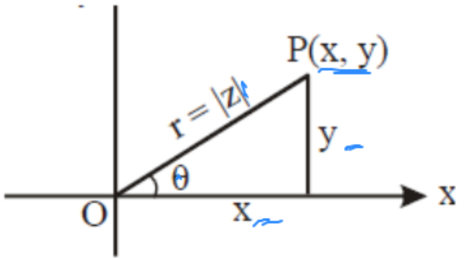
$Re(z^2) = x^2-y^2$

$x^2-y^2=0$

$\boxed{y=\pm x}$ pair of st lines.

REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

(b) Trignometric / Polar Representation :



Let the given complex number be $z = x + iy$

r and θ be the modulus and amp (z) respectively.

From the figure $x = r\cos\theta$, $y = r\sin\theta$

$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta) = r \text{ cis } \theta$

Hence, $z = r(\cos\theta + i\sin\theta)$ is called polar form of the complex number .

$x = r\cos\theta$

$y = r\sin\theta$

REPRESENTATION OF A COMPLEX NUMBER IN VARIOUS FORMS :

** $\cos\theta + i\sin\theta = e^{i\theta}$

(c) Exponential Representation :

$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$z = re^{i\theta}$ is called exponential form of the complex number.

where r is modulus of z and θ is amplitude of z.

Here, $\cos\theta + i\sin\theta = e^{i\theta} \ \forall\ \theta$

$Z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}.$

$z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$

$= \arg z_1 - \arg z_2$

$\left(\frac{z_1}{z_2}\right) = \frac{r_1}{r_2}\frac{e^{i\theta_1}}{e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right)e^{i(\theta_1 - \theta_2)}$

DEMOIVRE'S THEOREM (DMT) :

Case-I :

Statement :

If n is any integer then

(i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)\ldots\ldots(\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots\theta_n) + i\sin(\theta_1 + \theta_2 + \theta_3 + \ldots\ldots\theta_n)$

i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$(e^{i\theta})^n = e^{in\theta}$

ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) - - -$

$e^{i\theta_1} \cdot e^{i\theta_2} \cdot e^{i\theta_3} - - - = e^{i(\theta_1+\theta_2+\theta_3 - -)}$

Problems

$\frac{\pi}{2}+\frac{\pi}{2^2}$ —— ∞ $= \frac{a}{1-r} = \frac{\pi/2}{1-1/2} = \frac{\pi/2}{1/2} = \pi,$

If $x_n = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right)$ then $x_1x_2x_3 \ldots\ldots\ldots \infty$ is equal to -

(A) -1 (B) 1 (C) 0 (D) ∞

$x_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$x_2 = \cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}$

$x_3 = \cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}$

$x_1 \times x_2 \times x_3$ —— ∞

$= \cos\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3} \text{——} \infty\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{2^2}+\frac{\pi}{2^3} \ldots \infty\right)$

$= \cos\pi + i\sin\pi$

$= -1$

DEMOIVRE'S THEOREM (DMT) :

Case-II :

Statement : If $p, q \in Z$ and $q \neq 0$ then

$$(\cos\theta + i\sin\theta)^{p/q} = \boxed{\cos\left(\frac{2k\pi + p\theta}{q}\right) + i\sin\left(\frac{2k\pi + p\theta}{q}\right)}$$

where $k = 0, 1, 2, 3, \ldots\ldots, q-1$

$\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}$

Note : When index 'n' is integer then $(\cos\theta + i\sin\theta)^n$ has exactly one value which is $\cos n\theta + i\sin n\theta$ but when n is rational number (say p/q, $q \neq 0$) other than integer then $(\cos\theta + i\sin\theta)^{p/q}$ has exactly q different values.

Cube Roots of unity :

$z = 1^{1/3}$

$\cos\left(\frac{2m\pi + 1\times 0}{3}\right) + i\sin\left(\frac{2m\pi + 1\times 0}{3}\right)$

$$z^3 - 1 = 0 \;\Rightarrow\; z = (1)^{1/3} = \boxed{(\cos 0 + i\sin 0)^{1/3}} = (\cos 2m\pi + i\sin 2m\pi)^{1/3}$$

$$= \cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3},\; m = 0, 1, 2$$

ω & ω^2 are roots of $x^2+x+1=0$ ω, ω^2

$\omega+\omega^2 = -1 \Rightarrow \boxed{1+\omega+\omega^2=0}$

$\omega\times\omega^2 = 1 \Rightarrow \boxed{\omega^3=1}$

$$m = 0,\; z_1 = \cos 0 + i\sin 0 = 1$$

$$m = 1,\; z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{-1}{2} + i\frac{\sqrt{3}}{2} = \omega$$

$1, \omega, \omega^2$

$x = 1^{1/3}$

$$m = 2,\; z_3 = \cos\frac{4\pi}{3}, + i\sin\frac{4\pi}{3} = \frac{-1}{2} - i\frac{\sqrt{3}}{2} = \omega^2$$

$x^3 = 1$

$x^3 - 1 = 0$

$(x-1)(x^2+x+1) = 0$

$x = 1$ ω ω^2.

The cube roots of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$.

$1+\omega+\omega^2 = 0$

Cube Roots of unity :

$i^{4n} = 1$

$\omega^{3n} = 1$

Cube Root of unity = 1 $(1,0)$

$\frac{-1+\sqrt{3}i}{2}$ $\frac{-1}{2}, \frac{\sqrt{3}}{2}$

$\frac{-1-\sqrt{3}i}{2}$ $\frac{-1}{2}, -\frac{\sqrt{3}}{2}$

Note :

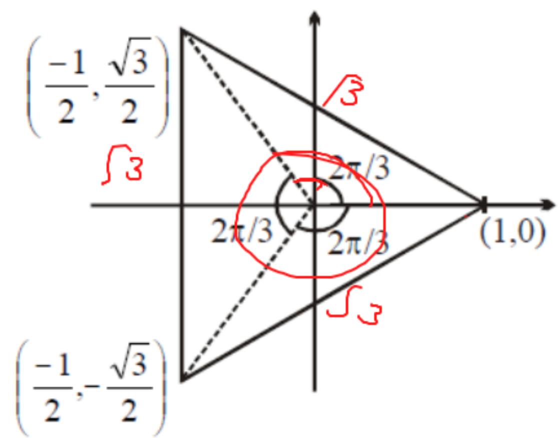
(i) $\omega^3 = 1$

(ii) $\omega^{3n} = 1, n \in I$

(iii) $1 + \omega + \omega^2 = 0$

(iv) $1 + \omega^r + \omega^{2r} = \begin{cases} 0 & \text{if r is not a multiple of 3.} \\ 3 & \text{if r is a multiple of 3.} \end{cases}$

(v) Representation of cube roots of unity on argand plane. Cube roots of unity form an equilateral Δ whose side is $\sqrt{3}$ units.



Problems

If α & β are imaginary cube roots of unity then $\alpha^n + \beta^n$ is equal to -

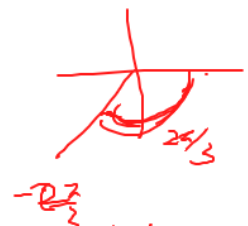
ω ω^2

(A) $2\cos\frac{2n\pi}{3}$ (B) $\cos\frac{2n\pi}{3}$ (C) $2i\sin\frac{2n\pi}{3}$ (D) $i\sin\frac{2n\pi}{3}$

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \alpha$$

$$\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$$

$$= \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = \beta$$



$$\alpha^n = \left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$$

$$\alpha^n + \beta^n = 2\cos\frac{2n\pi}{3}$$

$$\beta^n = \left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)^n = \cos\frac{2n\pi}{3} - i\sin\frac{2n\pi}{3}$$