Problem Solving on Definite Integration

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Easy

The value of $\int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x} dx$ is $= I$

(2019 Main, 9 April I)

(a) $\frac{\pi - 1}{2}$ (b) $\frac{\pi - 2}{8}$ (c) ✓ $\frac{\pi - 1}{4}$ (d) $\frac{\pi - 2}{4}$

$$\int_0^{\pi/2} \frac{\cos^3 x}{\cos x + \sin x} = I$$

$$\int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 2I$$

$$\frac{(\sin x + \cos x)(\overbrace{\sin^2 x + \cos^2 x}^{1} - \sin x \cos x)}{(\sin x + \cos x)}$$

$$\int_0^{\pi/2} (1 - \sin x \cos x)\, dx$$

$$\left(\frac{\pi}{2}\right) - \frac{1}{2}\int_0^{\pi/2} \sin 2x\, dx = 2I$$

$$\frac{\pi}{2} + \frac{\cos 2x}{4}\Big|_0^{\pi/2}$$

$$\frac{\pi}{2} + \left(\frac{-1}{4} - \frac{1}{4}\right) = 2I$$

$$\frac{\pi}{2} - \frac{1}{2} = \boxed{2I}$$

$$\Rightarrow \boxed{I = \frac{\pi - 1}{4}}$$

3 ★ If $\int_0^{\pi/2} \frac{\cot x}{\cot x + \operatorname{cosec} x} dx = m(\pi + n)$, then $m \cdot n$ is equal to **(2019 Main, 12 April I)**

(a) $-\frac{1}{2}$ (b) 1 (c) $\frac{1}{2}$ (d) -1

$$\int_0^{\pi/2} \left[\frac{\cos x + 1 - 1}{\cos x + 1}\right] dx$$

$$\int_0^{\pi/2} \left[1 - \frac{1}{1+\cos x}\right] dx$$

$$\int_0^{\pi/2} dx - \int_0^{\pi/2} \frac{1}{2\cos^2 x/2} dx$$

$$\frac{\pi}{2} - \frac{1}{2}\int_0^{\pi/2} \sec^2(x/2)\, dx \qquad \frac{x}{2} = t,\ dx = 2dt$$

$$-\frac{1}{2}\left(\tan\left(\frac{x}{2}\right) \cdot 2\right)\Big|_0^{\pi/2}$$

$$\left[\frac{\pi}{2} - 1\right] = m\pi + mn$$

$$m = \frac{1}{2}, \quad mn = -1$$

9) If $f(x)=\dfrac{2-x\cos x}{2+x\cos x}$ and $g(x)=\log_e x$, $(x>0)$ then the value of the integral $\int_{-\pi/4}^{\pi/4} g(f(x))dx$ is

(2019 Main, 8 April I)

(a) $\log_e 3$ (b) $\log_e e$

(c) $\log_e 2$ (d) $\log_e 1$ ✓

$$\int_{-\pi/4}^{\pi/4} g(f(x)) = \int_0^{\pi/4} [g(f(x)) + g(f(-x))]\,dx$$

$$g(f(x)) = \log_e\left(\frac{2-x\cos x}{2+x\cos x}\right)$$

$$g(f(-x)) = \log_e\left(\frac{2+x\cos x}{2-x\cos x}\right)$$

$$= -1\log_e\left(\frac{2-x\cos x}{2+x\cos x}\right)$$

9) The value of the integral $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\,dx$

(where, $[x]$ denotes the greatest integer less than or equal to x) is **(2019 Main, 11 Jan I**

(a) $4-\sin 4$ (b) 4

(c) $\sin 4$ (d) 0

$$= \int_0^2 \left(\frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} + \frac{\sin^2 x}{\left[-\frac{x}{\pi}\right]+\frac{1}{2}} \right)$$

$$[-x] = -[x] - 1$$

M-1

$$\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} = \int_{-2}^{0} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} + \int_{0}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}$$

$$-\frac{2}{\pi} < \frac{x}{\pi} < 0$$

$$= \int_{-2}^{0} -2\sin^2 x\,dx + \int_0^2 2\sin^2 x\,dx$$

$x = -t$

$$= \int_2^0 (+2\sin^2 t)\,dt + \int_0^2 2\sin^2 x\,dx$$

$$= -\int_0^2 2\sin^2 t\,dt + \int_0^2 2\sin^2 x\,dx$$

$$= 0$$

$$\int_0^2 \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} + \frac{\sin^2 x}{-\left[\frac{x}{\pi}\right]-\frac{1}{2}}$$

$$\frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} - \left(\frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\right)$$

0

The value of $\int_0^{\pi} |\cos x|^3 \, dx$ is

(2019 Main, 9 Jan I)

(a) $\frac{2}{3}$ (b) $-\frac{4}{3}$ (c) 0 (d) $\frac{4}{3}$

The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x}\,dx$ is

(2018 Main)

(a) $\frac{\pi}{8}$ (b) $\frac{\pi}{2}$ (c) 4π (d) $\frac{\pi}{4}$

The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$ is equal to **(2015, Main)**

(a) 2 (b) 4 (c) 1 (d) 6

If for a real number y, $[y]$ is the greatest integer less than or equal to y, then the value of the integral $\int_{\pi/2}^{3\pi/2} [2 \sin x]\, dx$ is **(1999, 2M)**

(a) $-\pi$ (b) 0

(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x] + [\sin x] + 4}$, where $[t]$ denotes the greatest integer less than or equal to t, is

(2019 Main, 10 Jan II)

(a) $\frac{1}{12}(7\pi - 5)$

(b) $\frac{1}{12}(7\pi + 5)$

(c) $\frac{3}{10}(4\pi - 3)$

(d) $\frac{3}{20}(4\pi - 3)$

If $\int_0^x f(t)\,dt = x^2 + \int_x^1 t^2 f(t)dt$, then $f'\left(\frac{1}{2}\right)$ is

(2019 Main, 10 Jan II)

(a) $\frac{24}{25}$ (b) $\frac{18}{25}$ (c) $\frac{6}{25}$ (d) $\frac{4}{5}$

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+\ldots+\frac{1}{5n}\right)$$ is equal to

(2019 Main, 12 Jan II)

(a) $\tan^{-1}(3)$ (b) $\tan^{-1}(2)$

(c) $\pi/4$ (d) $\pi/2$

$$\lim_{n \to \infty} \left(\frac{(n+1)^{1/3}}{n^{4/3}} + \frac{(n+2)^{1/3}}{n^{4/3}} + \ldots + \frac{(2n)^{1/3}}{n^{4/3}} \right)$$ is equal to

(2019 Main, 10 April I)

(a) $\frac{4}{3}(2)^{4/3}$

(b) $\frac{3}{4}(2)^{4/3} - \frac{4}{3}$

(c) $\frac{3}{4}(2)^{4/3} - \frac{3}{4}$

(d) $\frac{4}{3}(2)^{3/4}$

The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$ equals

(2019 Main, 11 Jan II)

(a) $\frac{1}{5}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)\right)$ (b) $\frac{1}{20}\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)$

(c) $\frac{1}{10}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$ (d) $\frac{\pi}{40}$

Let $I = \int_a^b (x^4 - 2x^2)\, dx$. If I is minimum, then the ordered pair (a, b) is **(2019 Main, 10 Jan I)**

(a) $(-\sqrt{2}, 0)$
(b) $(0, \sqrt{2})$
(c) $(\sqrt{2}, -\sqrt{2})$
(d) $(-\sqrt{2}, \sqrt{2})$

Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}}\,dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$. Then which one of the following is true ? **[AIEEE 2008]**

(A) $I < \frac{2}{3}$ and $J < 2$

(B) $I < \frac{2}{3}$ and $J > 2$

(C) $I > 2/3$ and $J < 2$

(D) $I > 2/3$ and $J > 2$