Permutation & Combination

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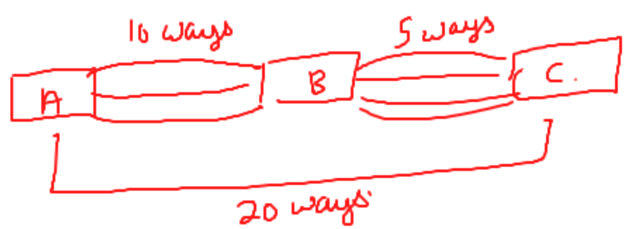
Concepts

FUNDAMENTAL PRINCIPLE OF COUNTING (counting without actual counting):

If an event A can occur in 'm' different ways and another event B can occur in 'n' different ways, then the total number of different ways of-

(a) simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(b) happening exactly one of the events is $m + n$ (known as addition principle).



$$\begin{aligned}\text{Total ways} &= (\text{via } B) + (\text{direct}) \\ &= 10 \times 5 + 20 \\ &= 50 + 20 \\ &= 70.\end{aligned}$$

Problems

How many 3 digit numbers can be formed by the digit 1, 2, 3, 4, 5 without repetition.

$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$
5 4 3 $= 5 \times 4 \times 3 = 60$

Problems

How many diffrent words can be formed using all the letters in the word "MIRACLE".

(a) *If vowels may accupy the even position.*

(b) *In the above case if vowels may occupy odd position.*

Consonants: M, R, C, L   Vowels. I, A, E

a)

4 ↑ 3 ↑ 2 ↑ 1

  3   2   1

$= (4\times3\times2\times1)\times(3\times2\times1)$

$= 24\times6 = 144.$

b)

↑ 4 ↑ 3 ↑ 2 ↑

4   3   2   1

$= (4\times3\times2)\times(4\times3\times2\times1)$

$= 24\times24$

$= 576$

Permutation

Permutation means arrangement in a definte order of things which may be alike or different taken some or all at a time. Hence permutation refers to the situation where order of occurrence of the events is important.

OR

Permutation is arrangement of things in same definite order

If ${}^nP_r$ denotes the number of permutations of n different things, taking r at a time, then ${}^nP_r = n(n-1)(n-2).....(n-r+1) = \frac{n!}{(n-r)!}$ Note that, ${}^nP_n = n!$.

$${}^nP_r = \frac{n!}{(n-r)!}$$

Note :

(i) ${}^nP_n = n!$, ${}^nP_0 = 1$, ${}^nP_1 = n$

(ii) Number of arrangements of n distinct things taken all at a time = n!

n places

n n-1 n-2 n-3 -- -- 1

$n \times n-1 \times n-2$ —— 1 $= n!$

Combination

Combination refers to the situation where order of occurrence of the event is not important.
or combination is selection of one or more things out of n things.

If ^nC_r denotes the number of ~~combinations~~ selections of n different things taken r at a time, then

$$\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}} = \frac{{}^nP_r}{r!} \text{ where } r \le n \;;\; n \in N \text{ and } r \in W.$$

Important Results :

(i) $^nC_0 = 1$

(ii) $^nC_n = 1.$

(iii) $^nC_{n-r} = {}^nC_r \quad \Rightarrow \quad {}^nC_x = {}^nC_y \quad \Rightarrow \quad x + y = n \quad$ or $\quad x = y$

(iv) $\boxed{{}^nP_r = {}^nC_r\, r!}$

i.e. permutation is defined total number of combination of object then arragement of objects.

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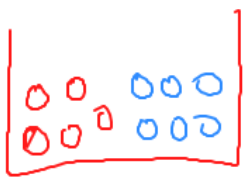
Problems

A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be drawn so that there are atleast two balls of each colour ?



2R 4W + 3R 3W + 4R 2W

$$\boxed{{}^5C_2 \times {}^6C_4 + {}^5C_3 \times {}^6C_3 + {}^5C_4 \times {}^6C_2}$$

Problems

A B C D

On a new year day, every students of a class sends a card to every other student, the postman delivers 600 cards. How many students are there in class?

B C A D

Total students $= n$

$2 \times {}^{n}C_2 = 600$

${}^{n}C_2 = 300$

$\frac{n(n-1)}{2} = \cancel{600}\ 300$

$n(n-1) = 600$

$\boxed{n=25}$

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$A \; B \qquad C \; D$

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$\boxed{n = 25}$

Problems

4 Boys & 4 Girls are to be seated in a line find number of ways
They can be seated so that "No two girls are together"

→ Girls should be present in the gap of boys

Gap Method

$B_1\ B_2\ B_3\ B_4.$ $\quad G_1\ G_2\ G_3\ G_4.$

$\{ \ \_ \ \{ \ \_ \ \{ \ \_ \ \{ \ \_ \ \{$

$4! \times {}^5C_4 \times 4! = 24 \times 5 \times 24$

$= 576 \times 5$

Formation of Groups

$m+n \to m,\ n$ $\quad {}^{m+n}C_m = \frac{(m+n)!}{m!\,n!}$

(a) (i) The number of ways in which (m + n) different things can be divided into two groups such that one of them contains m things and other has n things, is $\boxed{\dfrac{(m+n)!}{m!\;n!}}$ (m ≠ n).

(ii) If m = n, it means the groups are equal & in this case the number of divisions is $\boxed{\dfrac{(2n)!}{n!\;n!\;2!}}$. As in any one way it is possible to interchange the two groups without obtaining a new distribution.

$2n \to n,\ n$ $\qquad \frac{{}^{2n}C_n}{2!} = \frac{2n!}{n!\,n!\,2!}$ $\qquad 10 \to 5,\ 3,\ 2$ $\qquad = \frac{10!}{5!\,3!\,2!}$

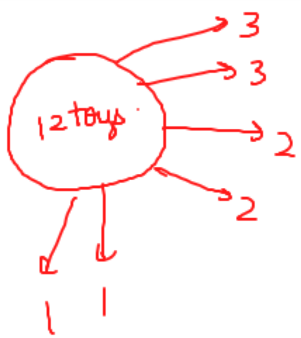
(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is : $\dfrac{(m+n+p)!}{m!\;n!\;p!}$, m ≠ n ≠ p.

(ii) If m = n = p then the number of groups = $\boxed{\dfrac{(3n)!}{n!\;n!\;n!\;3!}}$.

$10 \to 3,\ 3,\ 2,\ 2$ $\qquad \frac{10!}{3!\,3!\,2!\,2!\,2!\times 2!}$

Problems

In how many ways 6 bundles of 12 different toys be made such that 2 bundles are of 3 toys each, 2 bundles are 2 toys each & 2 bundle of 1 toy each



$$\frac{12!}{3!\,3!\;2!\,2!\;1!\,1! \times 2! \times 2! \times 2!}$$

Problems

Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

P_1 P_2 P_3 P_4

A_1 A_2 A_3 A_4

48 cards → 12, 12, 12, 12

$$\frac{48!}{12!\,12!\,12!\,12!\,4!}$$

Ans: $\boxed{\underbrace{\frac{48!}{(12!)^4 \times 4!}}_{\text{Group Formation}} \times 4! \times 4!}$

Arrangement of bundles

Arrangement of Aces

Permutation of Alike Objects

The number of permutations of n things taken all at a time when p of them are similar & of one type, q of them are similar & of another type, r of them are similar & of a third type & the remaining $n-(p+q+r)$ are all different is : $\frac{n!}{p!q!r!}$.

MIRACLE

IITJEE

$\frac{6!}{2!\,2!}$

_ _ _ _ _ _ _

7 6 5 4 3 2 1

7!

Problems

Find total number of word's formed by using all letters of the word "IITJEE".

$$\frac{6!}{2!\,2!} = \frac{720}{2\times 2} = \frac{\cancel{720}^{180}}{\cancel{4}} = 180$$

Problems

Consider word *ASSASSINATION*, *find number of ways of arranging the letters.*

~~*(i)*~~ *Number of words using all.*

Gap *(ii)* *If no two vowels are together.*

~~*(iii)*~~ *If all S are seperated.*

SSSS NN T

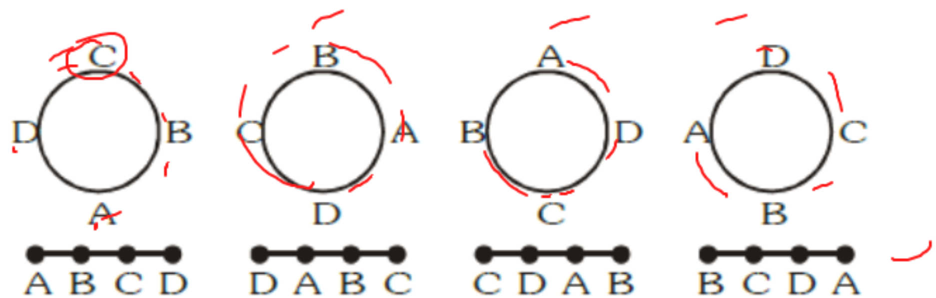
AAA II O

i) $\frac{13!}{4!\,3!\,2!\,2!}$

ii) $\frac{7!}{4!\,2!} \times {}^{8}C_{6} \times \frac{6!}{3! \times 2!}$

iii) $\frac{9!}{3!\,2!\,2!} \times {}^{10}C_{4} \times \frac{4!}{4!}$

Circular Permutation



The number of circular permutations of n different things taken all at a time is ; $(n-1)!$. If clockwise & anti–clockwise circular permutations are considered to be same, then it is $\frac{(n-1)!}{2}$.

Note : Number of circular permutations of n things when p alike and the rest different taken all at a time distinguishing clockwise and anticlockwise arrangement is $\frac{(n-1)!}{p!}$.

Problems

In how many ways 7 diffrent flowers can be formed into a garland.

A B C D E F G

Ans: $\frac{(n-1)!}{2} = \frac{(7-1)!}{2} = \frac{6!}{2} = \frac{720}{2} = 360$

Problems

Find number of ways in which 7 American and 7 British people can be seated on a round table so that no two Americans are consecutive.

$(7-1)! \times 7! = 6! \times 7!$

Total No of Selections

Given n different objects, the number of ways of selecting atleast one of them is , ${}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 + \ldots + {}^{n}C_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

$${}^{n}C_1 + {}^{n}C_2 + {}^{n}C_3 - - {}^{n}C_n = 2^n - 1$$

Total number of ways in which it is possible to make a selection by taking some or all out of $p + q + r + \ldots$ things , where p are alike of one kind, q alike of a second kind , r alike of third kind & so on is given by : $(p + 1)(q + 1)(r + 1) \ldots - 1$.

AAAA

0A	1A	2A	3A	4A
1	1	1	1	1

Problems

Find the number of ways in which one or more letter be selected from the letters "AAAABBCCCDEF"

$(4+1)\ (2+1)\ (3+1)\ (1+1)\ (1+1)\ (1+1) - 1$

$5 \times 3 \times 4 \times 2 \times 2 \times 2 - 1$

Total Distribution

(a) **Distribution of distinct objects :** Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by : p^n

25

$5 \times 5 \times 5 \times 5 \ldots = 5^{25}$

(b) **Distribution of alike objects :** Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

Beggars

25 alike coins

$\boxed{{}^{25+5-1}C_{5-1} = {}^{29}C_4}$

Problems

Find total number of ways of distributing 7 identical computers to R|S|G. So that each receive atleast one computer

1 1 1

Give 1 Computer each to all three.

4 computers. 3 people.

0000

$$ {}^{n+p-1}C_{p-1} = {}^{4+3-1}C_{3-1} = {}^{6}C_{2} $$

Dearrangement

Number of ways in which n letters can be placed in n directed letters so that no letter goes into its own envelope is $= n!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} + \dots\dots\dots\dots + (-1)^n \frac{1}{n!}\right]$. ✓

$L_1 \quad L_2 \quad L_3$

$E_1 \quad E_2 \quad E_3$

L_1	L_2	L_3	×
L_1	L_3	L_2	×
L_2	L_1	L_3	×
L_2	L_3	L_1	✓
L_3	L_1	L_2	✓
L_3	L_2	L_1	×

$D_3 = 2$

$D_4 = 4!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$