PHYSICS

JEE and NEET CRASH COURSE

Rotational Motion

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An Initiative by अमर उजाला

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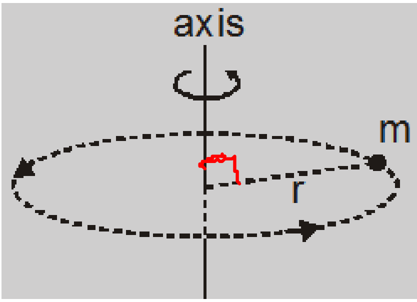
Moment of Inertia

The moment of inertia of a particle with respect to an axis of rotation is equal to the product of mass of the particle and square of distance from rotational axis.

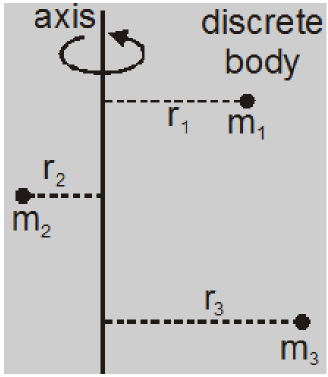
$$I = mr^2$$

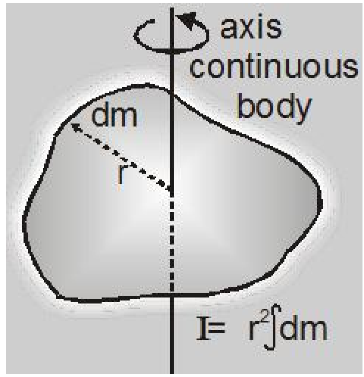


Moment of inertia of system of particle

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots$$

$$I = \sum mr^2$$

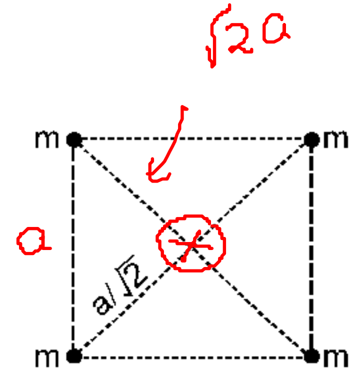




$$I = \int dm\, r^2$$

Example

Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.



Sol.

The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2} ma^2$. The moment of inertia of the system is, therefore,

$$4 \times \frac{1}{2} ma^2 = 2ma^2.$$

$$I = m\left(\frac{a}{\sqrt{2}}\right)^2 \times 4$$

$$= m\frac{a^2}{2} \times 4$$

$$= 2ma^2$$

Radius of gyration

$$I = MK^2$$

$$2ma^2 = 4mK^2$$

$$K = \frac{a}{\sqrt{2}}$$

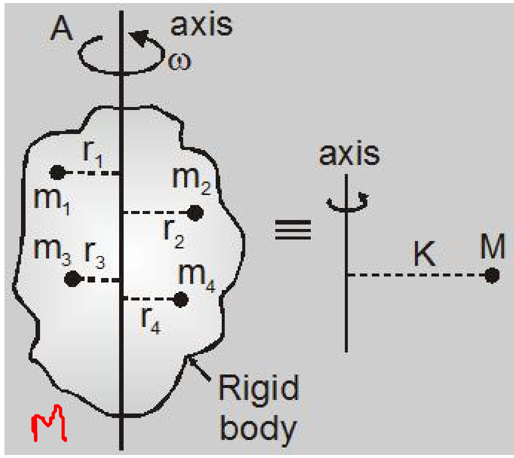
Radius of Gyration (K)

The radius of gyration of a body is the distance from axis of rotation, the square of this distance when multiplied by the mass of body then it gives the moment of inertia of the body ($I = MK^2$) about same axis of rotation.

$$I = MK^2$$

Radius of gyaration $$K = \sqrt{\frac{I}{M}}$$

where,
I = Moment of inertia of system about the axis, and
M = Total mass of the system



Theorems of moment of Inertia

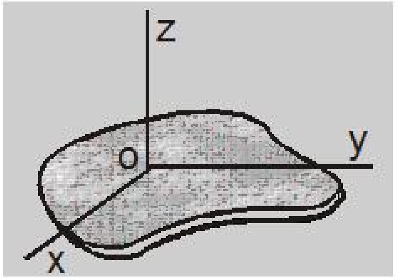
(applicable only for two dimensional bodies or plane laminas)

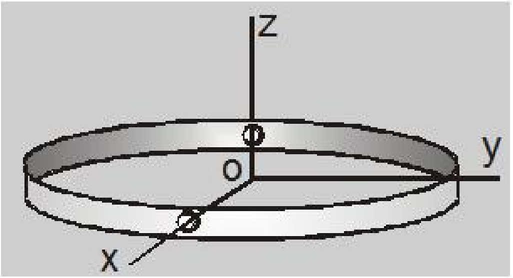
$$I_z = I_x + I_y$$

where I_x = MI of the body about X-axis

I_y = MI of the body about Y-axis

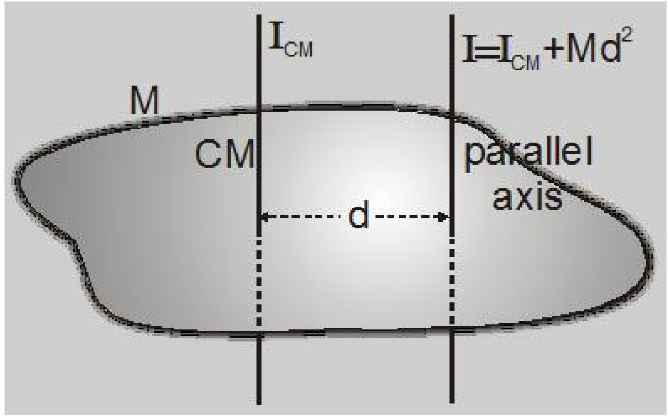
I_z = MI of the body about Z-axis





Theorems of moment of Inertia

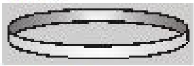
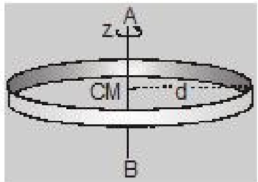
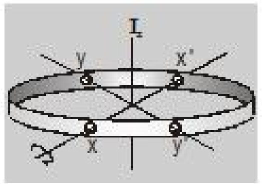
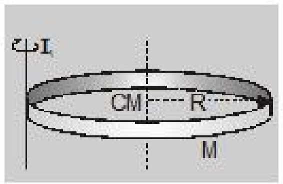
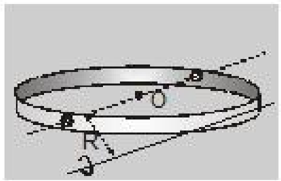
______________________ (applicable for all types of bodies)



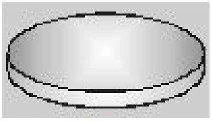
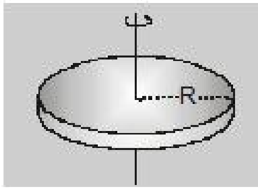
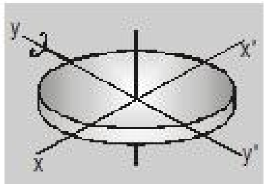
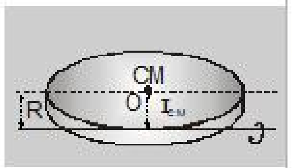
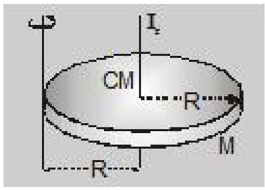
$$I = I_{CM} + Md^2$$

I_{CM} = Moment of inertia about the axis passing through centre of mass

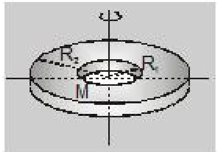
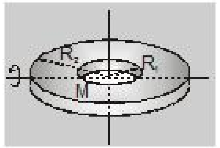
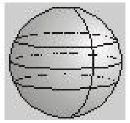
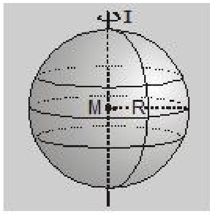
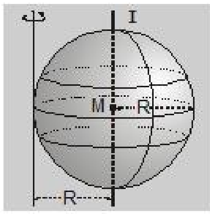
Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(1) Circular Ring  Mass = M Radius = R	(a) About an axis perpendicular to the plane and passes through the centre		MR^2	R
	(b) About the diametric axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(c) About an axis tengential to the rim and perpendicular to the plane of the ring		$2MR^2$	$\sqrt{2}\,R$
	(d) About an axis tangential to the rim and lying in the plane of ring		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}\,R$

Moment of inertia of some regular bodies

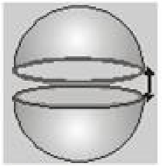
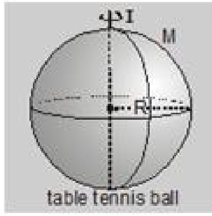
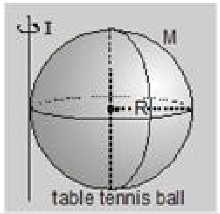
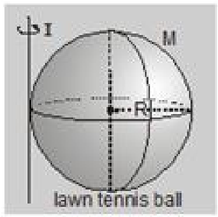
Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(2) Circular Disc  Mass = M Radius = R	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(b) About a diametric axis		$\frac{MR^2}{4}$	$\frac{R}{2}$
	(c) About an axis tangential to the rim and lying in the plane of the disc		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$
	(d) About an axis tangential to the rim & perpendicular to the plane of disc		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$

Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(3) Annular disc  M = Mass R_1 = Internal Radius R_2 = Outer Radius ($R_2 > R_1$)	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{M}{2}\left[R_1^2+R_2^2\right]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
	(b) About a diametric axis		$\frac{M}{4}\left[R_1^2+R_2^2\right]$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
(4) Solid Sphere  M = Mass R = Radius	(a) About its diametric axis which passes through its centre of mass		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
	(b) About a tangent to the Sphere		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$

$\frac{M}{2}(R_1^2+R_2^2)$

Moment of inertia of some regular bodies

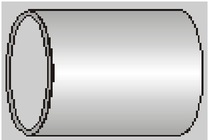
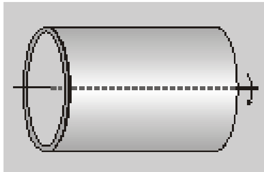
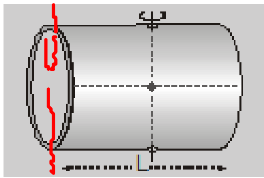
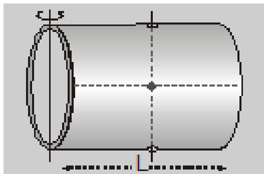
Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(5) Hollow Sphere (Thin spherical Shell)  M = Mass R = Radius Thickness negligible	(a) About diametric axis passing through centre of mass		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
	(b) About a tangent to the surface		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
(6) Hollow sphere with cavity  r=Internal radius R=Outer radius M=Mass	About diametric axis passes through centre of mass		$\frac{2}{5}M\frac{(R^5-r^5)}{(R^3-r^3)}$	$\sqrt{\frac{2}{5}\frac{(R^5-r^5)}{(R^3-r^3)}}$

$I' = \frac{2}{5}MR^2$

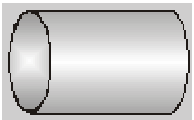
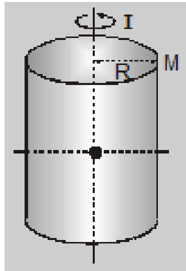
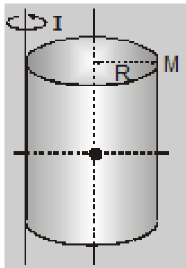
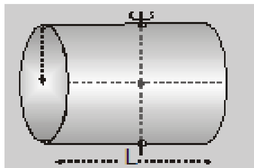
$I' = \frac{2}{5}MR^2 + MR^2$

$= \frac{7}{5}MR^2$

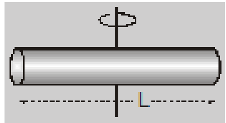
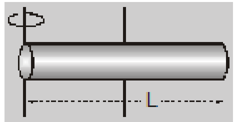
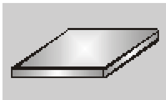
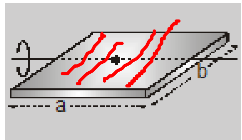
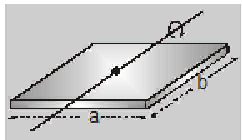
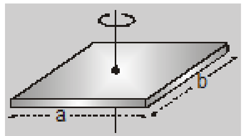
Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(7) Hollow Cylinder  M = Mass R = Radius L = Length	(a) About its geometrical axis which is parallel to its length		MR^2	R
	(b) About an axis which is perpendicular to its length and passes through its centre of mass		$M\left[\frac{R^2}{2}+\frac{L^2}{12}\right]$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{12}}$
	(c) About an axis perpendicular to its length and passing through one end of the cylinder		$M\left[\frac{R^2}{2}+\frac{L^2}{3}\right]$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{3}}$

Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(8) Solid Cylinder M = Mass R = Radius L = Length 	(a) About its geometrical axis, which is parallel to its length		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	(b) About an axis tangential to the cylinderical surface and parallel to its geometrical axis		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
	(c) About an axis passing through the centre of mass and perpendicular to its length		$M\left[\frac{L^2}{12}+\frac{R^2}{4}\right]$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{4}}$

Moment of inertia of some regular bodies

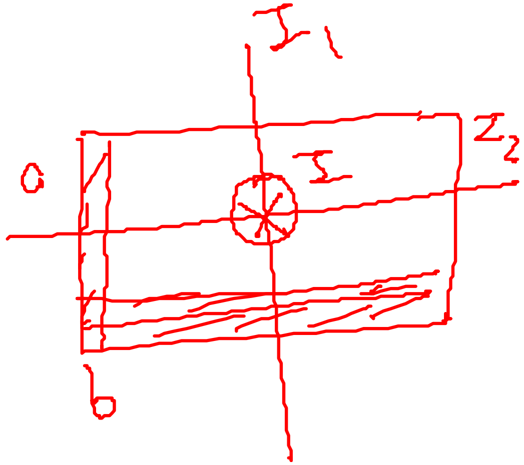
Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(9) Thin Rod 	(a) About an axis passing through centre of mass and perpendicular to its length		$\frac{ML^2}{12}$	$\frac{L}{2\sqrt{3}}$
Thickness is negligible w.r.t. length	(b) About an axis passing through one end and perpendicular to length of the rod		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
(10) Rectangular Plate 	(a) About an axis passing through centre of mass and perpendicular to side b in its plane		$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
M = Mass a = Length b = Breadth	(b) About an axis passing through centre of mass and perpendicular to side a in its plane.		$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$
	(c) About an axis passing throught centre of mass and perpendicular to plane		$\frac{M}{12}(a^2+b^2)$	$\sqrt{\frac{a^2+b^2}{12}}$

$I_1 = \frac{Mb^2}{12}$

$I_2 = \frac{Ma^2}{12}$

$I = I_1 + I_2$

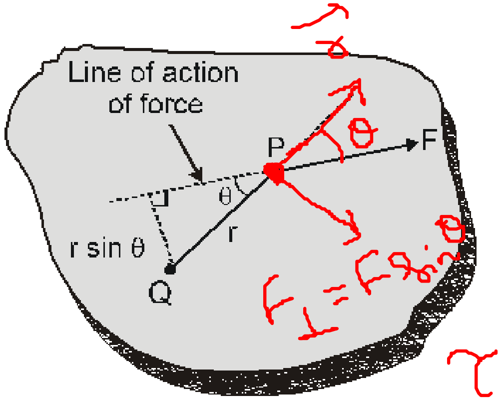
$= \frac{M}{12}(a^2+b^2)$



$\vec{A}\times\vec{B} \neq \vec{B}\times\vec{A}$

Torque

Torque represents the capability of a force to produce change in the rotational motion of the body.



$\tau = (r\sin\theta)F$

$\tau = r(F\sin\theta)$

6.1 Torque about a point :

Torque of force $\vec{F}$ about a point $\vec{\tau} = \vec{r} \times \vec{F}$

Where $\vec{F}$ = force applied

P = point of application of force

Q = Point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force w.r.t. the point about which we want to determine the torque.

$$|\vec{\tau}| = rF\sin\theta = r_{\perp}F = rF_{\perp}$$

Where θ = angle between the direction of force and the position vector of P wrt. Q.

$r_{\perp} = r\sin\theta$ = perpendicular distance of line of action of force from point Q ,it is also called force arm.

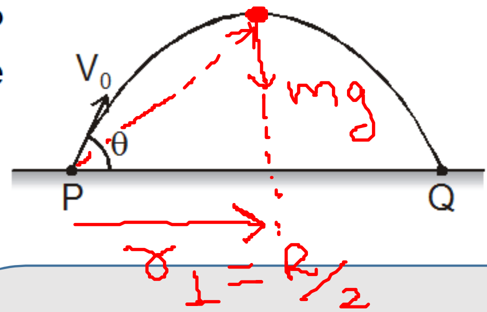
$F_{\perp} = F\sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$

SI unit of torque is Nm

$[ML^2T^{-2}]$

Example

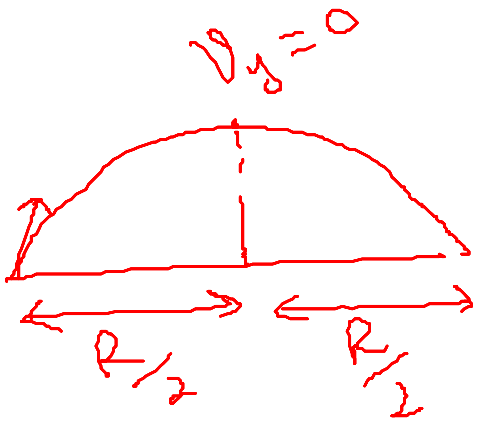
A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with horizontal. Find out the torque about the point of projection acting on the particle when it is at its maximum height ?



Sol.

$$\tau = rF\sin\theta = \frac{R}{2}\,mg = \frac{v_0^2\sin 2\theta}{2g}\,mg$$

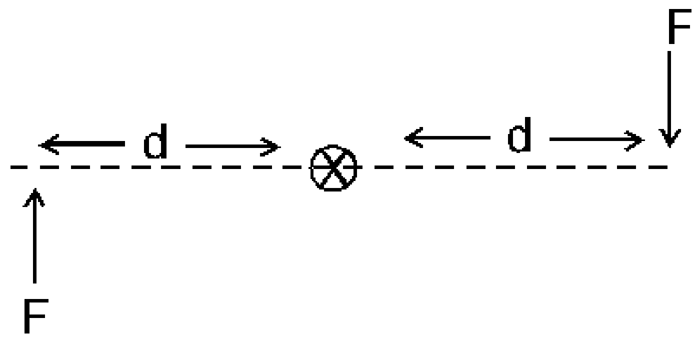
$$\tau = \frac{mv_0^2\sin 2\theta}{2}$$



$$\tau = r_\perp F = \frac{R}{2}\cdot mg = \frac{v_0^2\sin 2\theta}{2g}\cdot mg$$

Force Couple

A pair of forces each of same magnitude and acting in opposite direction is called a force couple.
Torque due to couple = Magnitude of one force × distance between their lines of action.
Magnitude of torque = $\tau = F\,(2d)$

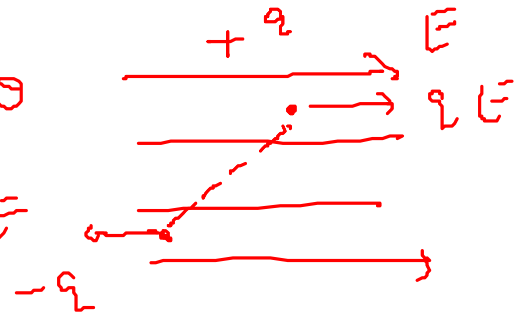


A couple does not exert a net force on an object even though it exerts a torque.

Net torque due to a force couple is same about any point.

$F_{net} = 0$

$\tau_{net} = pE\sin\theta$

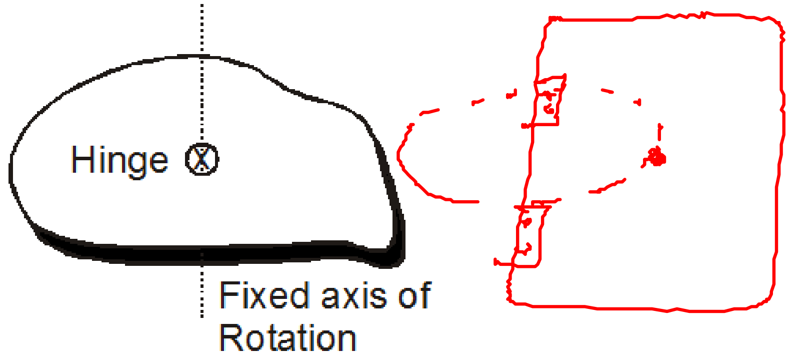


Rotation about a fixed axis

If I_{Hinge} = moment of inertia about the axis of rotation (since this axis passes through the hinge, hence the name I_{Hinge}).

$\left.\vec{\tau}_{ext}\right)$ = resultant external torque acting on the body about axis of rotation

α = angular acceleration of the body.



$$\left.\vec{\tau}_{ext}\right)_{Hinge} = I_{Hinge}\ \vec{\alpha}$$

Rotational Kinetic Energy = $\frac{1}{2}.I.\omega^2$

$$\vec{P} = M\vec{v}_{CM}$$

$$\vec{F}_{external} = M\vec{a}_{CM}$$

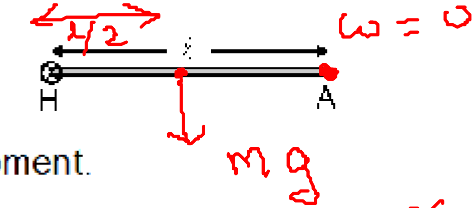
Net external force acting on the body has two component tangential and centripetal.

$$\Rightarrow \qquad F_C = ma_C = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM}$$

$$\Rightarrow \qquad F_t = ma_t = m\alpha\, r_{CM}$$

Example

A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.



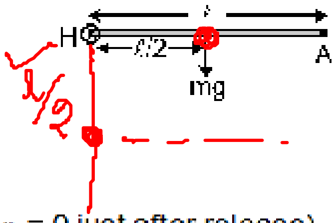
(i) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest ?

(ii) Calculate the acceleration (tangential and radial) of point A at this moment.

(iii) Find α and ω when rod becomes vertical.

Sol. (i) $\tau_H = I_H \alpha$

$$mg.\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \rightarrow \alpha = \frac{3g}{2\ell}$$



(ii) $a_{t} = \alpha\ell = \frac{3g}{2\ell}.\ell = \frac{3g}{2}$

$a_{cn} = \omega^2 r = 0.\ell = 0$ ($\because$ $\omega = 0$ just after release)

(iii) Torque = 0 when rod becomes vertical.

so $\alpha = 0$

using energy conservation $\frac{mg\ell}{2} = \frac{1}{2}I\omega^2 \quad \left(I = \frac{m\ell^2}{3}\right)$

$$\omega = \sqrt{\frac{3g}{\ell}}$$

$\tau_H = I_H \alpha \Rightarrow mg\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha$

$\alpha = 3g/2\ell$

$a_t = \alpha r = \frac{3g}{2\ell} \times \ell = 3g/2$

$a_r = \omega^2 r = 0 \times \ell = 0$

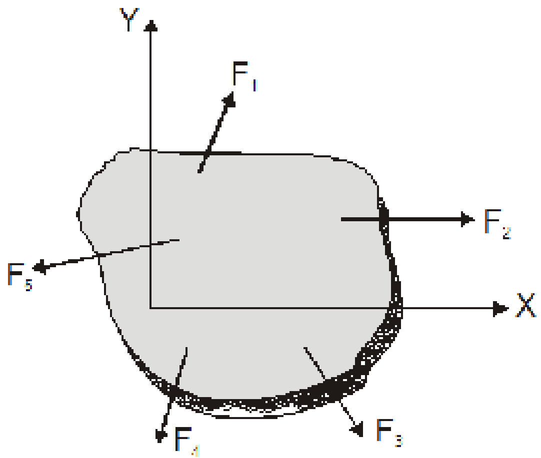
$\tau_H = 0 \Rightarrow \alpha = 0$

Equilibrium

A system is in mechanical equilibrium if it is in translational as well as rotational equilibrium.

For this : $\overline{F_{net}} = 0$

$\overline{\tau_{net}}$ 0 (about every point)



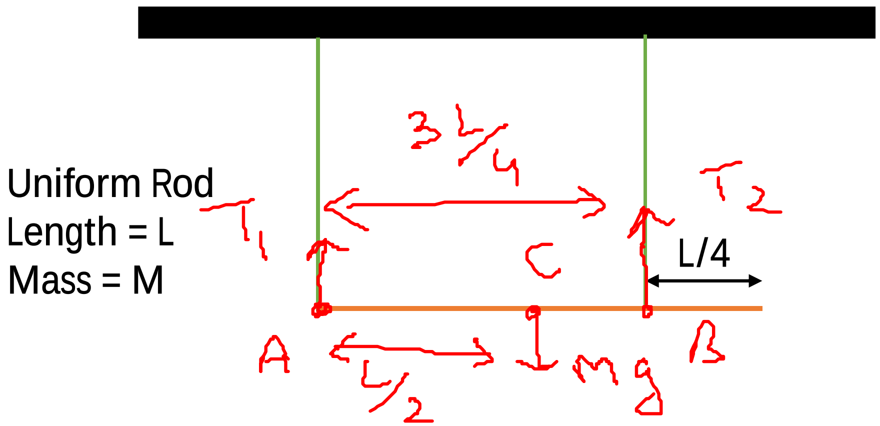
From (6.3), if $\overline{F_{net}}$ 0 then $\overline{\tau_{net}}$ is same about every point

Hence necessary and sufficient condition for equilibrium is $\overline{F_{net}} = 0$, $\overline{\tau_{net}} = 0$ about any one point,

which we can choose as per our convenience. ($\overline{\tau_{net}}$ will automatically be zero about every point)

Example

In the figure shown, find tension in both the strings if rod is in equilibrium.



Sol.

$$F_{net} = 0$$
$$T_1 + T_2 = Mg$$
$$\tau_A = 0$$

$$0 + Mg\frac{L}{2} - T_2\frac{3L}{4} = 0$$
$$\Rightarrow T_2 = \frac{2Mg}{3}$$

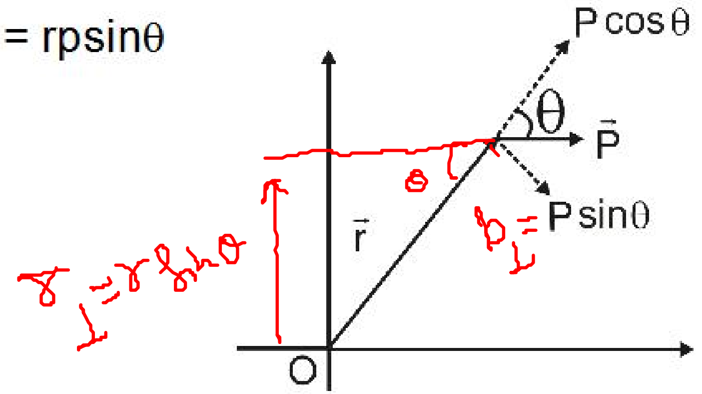
$$T_1 = Mg - T_2$$
$$= Mg - \frac{2Mg}{3}$$
$$T_1 = \frac{Mg}{3}$$

Angular Momentum

$\vec{L} = \vec{r} \times \vec{P}$ $\Rightarrow$ $L = rp\sin\theta$

or $|\vec{L}| = r_{\perp} \times P$

or $|\vec{L}| = P_{\perp} \times r$



where $\vec{P}$ = momentum of particle

$\vec{r}$ = position of vector of particle with respect to point O about which angular momentum is to be calculated .

θ = angle between vectors $\vec{r}$ & $\vec{P}$

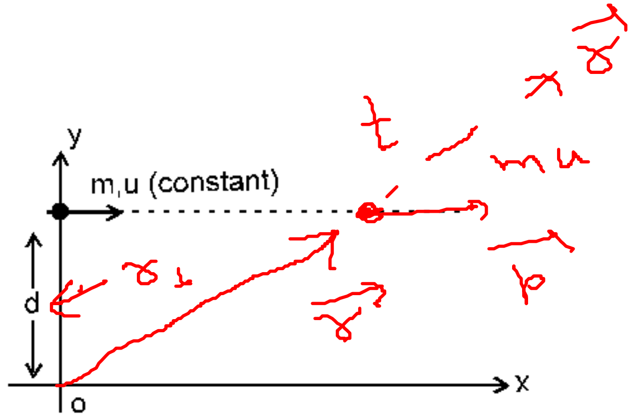
$r_{\perp}$ = perpendicular distance of line of motion of particle from point O.

$P_{\perp}$ = component of momentum perpendicular to $\vec{r}$.

SI unit of angular momentum is kgm^2/sec.

Example

A particle of mass 'm' starts moving from point (o,d) with a constant velocity $u\hat{i}$. Find out its angular momentum about origin at this moment what will be the answer at the later time?



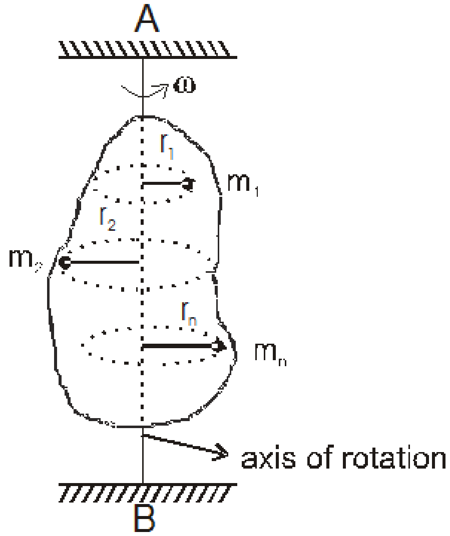
Sol.

$$\vec{L} = \vec{r} \times \vec{p} = mud \otimes$$

$$L = r_{\perp} p$$

$$= d \cdot mu$$

Angular momentum of a rigid body rotating about fixed axis



For fixed axis rotation;

$$K.E. = \frac{1}{2} I \omega^2$$

$$= \frac{L^2}{2I}$$

L = I ω

L = Angular momentum of object about axis of rotation.

I = Moment of Inertia of rigid , body about axis of rotation.

ω = angular velocity of the object.

Conservation of Angular Momentum

Newton's 2nd law in rotation : $\vec{\tau} = \frac{d\vec{L}}{dt}$

where $\vec{\tau}$ and $\vec{L}$ are about the same axis.

If $\tau_{ext} = 0$ about the axis of rotation.

then $\vec{L}$ = constant

Impulse of Torque : $\int \tau dt = \Delta L$

$\Delta L \rightarrow$ Charge in angular momentum.

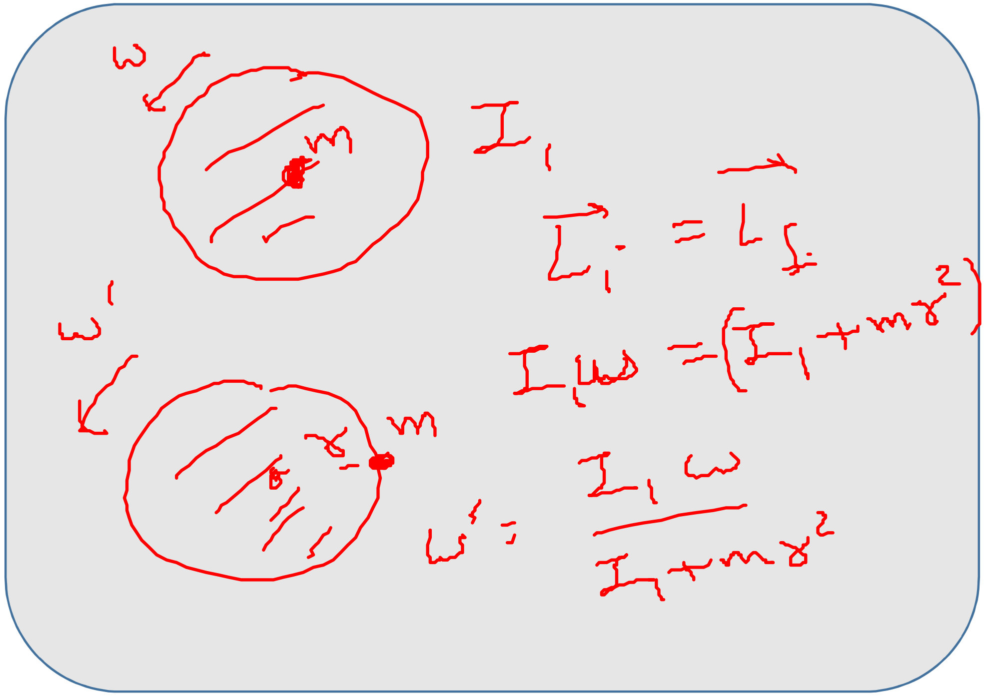
Example

A rotating table has angular velocity ' ω ' and moment of inertia I_1. A person of mass 'm' stands on centre of rotating table. If the person moves a distance r along its radius then what will be the final angular velocity of rotating table.

Sol.

Initial angular momentum = Final angular momentum

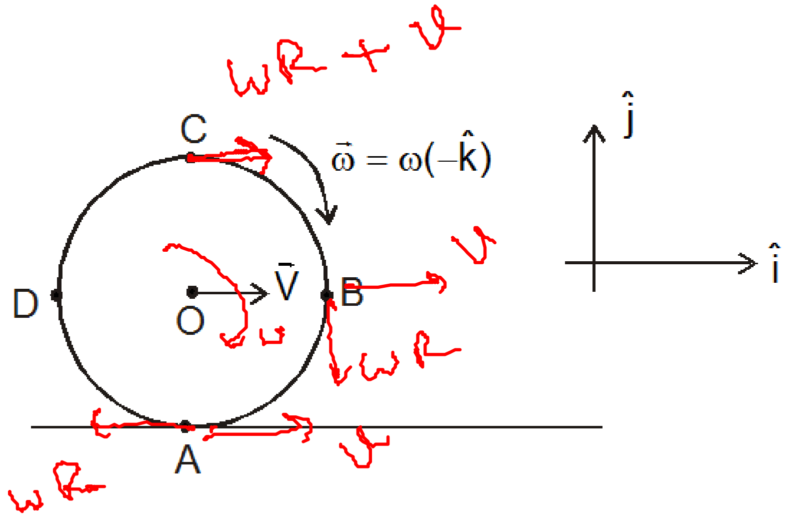
$$I_1\omega_1 = (I_1 + m r^2)\,\omega_2 \quad \Rightarrow \quad \omega_2 = \frac{I_1\omega_1}{I_1 + m r^2}$$



Combined Translational and Rotational motion of a rigid body

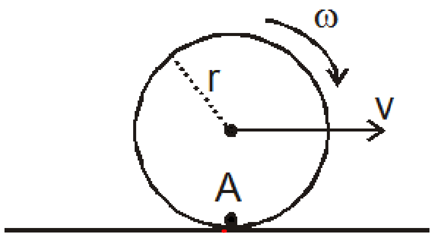
The general motion of a rigid body can be thought of as a sum of two independent motions. A translation of some point of the body plus a rotation about this point. A most convenient choice of the point is the centre of mass of the body as it greatly simplifies the calculations.

Consider the general motion of a wheel (radius r) which can be view on pure translation of its center O (with the velocity v) and pure rotation about O (with angular velocity ω)



Pure Rolling (or rolling without sliding)

Pure rolling means there is no relative motion between the rolling body and the surface of contact, at the point of contact.

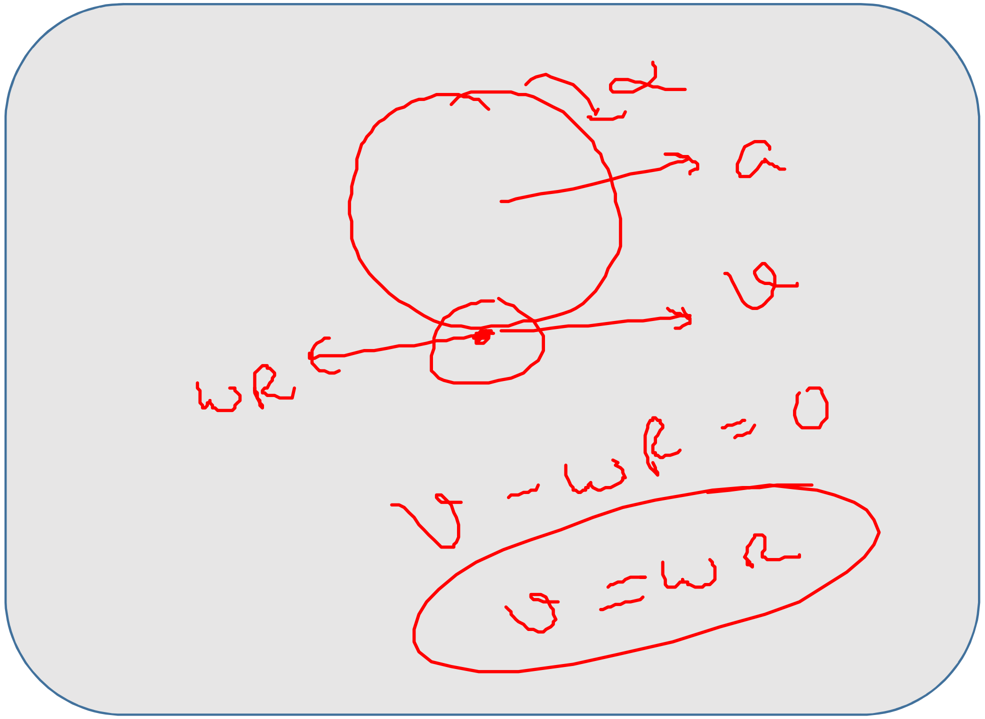


For pure rolling, velocity of A w.r.t. ground is zero.

$$\Rightarrow v - \omega r = 0$$

$$v = \omega r$$

Similarly $\quad a = \alpha r$



Dynamics of general motion of a rigid body

This motion can be viewed as translation of centre of mass and rotation about an axis passing through centre of mass

If

I_{CM} = Moment of inertia about this axis passing through COM

τ_{cm} = Net torque about this axis passing through COM

$\vec{a}_{CM}$ = Acceleration of COM

$\vec{v}_{CM}$ = Velocity of COM

$\vec{F}_{ext}$ = Net external force acting on the system.

$\vec{P}_{system}$ = Linear momentum of system.

$\vec{L}_{CM}$ = Angular momentum about centre of mass.

$\vec{r}_{CM}$ = Position vector of COM w.r.t. point A.

then

(i) $\vec{\tau}_{cm} = I_{cm}\vec{\alpha}$

(ii) $\vec{F}_{ext} = M\vec{a}_{cm}$

(iii) $\vec{P}_{system} = M\vec{v}_{cm}$

Tr. K.E   R.K.E

(vi) Total K.E.= $\frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$

(v) $\vec{L}_{CM} = I_{CM}\vec{\omega}$

(vi) Angular momentum about point A

= $\vec{L}$ about C.M. + $\vec{L}$ of C.M. about A

$$\vec{L}_A = I_{cm}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}$$

Example

A spherical ball rolls on a table without slipping. Then the fraction of its total energy associated with rotation is -

(A) 2/5 (B) 2/7 (C) 3/5 (D) 3/7

Sol. Total energy

$$K = K_R + K_T = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

$$= \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2 + \frac{1}{2} mr^2\omega^2$$

$$= \frac{1}{5} mr^2\omega^2 + \frac{1}{2} mr^2\omega^2 = \frac{7}{10} mr^2\omega^2$$

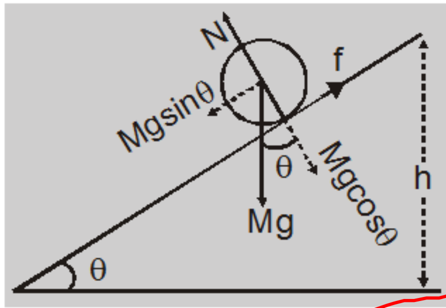
Now, rotational kinetic energy

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{5} mr^2\omega^2$$

$$\therefore \frac{K_R}{K} = \frac{\frac{1}{5}mr^2\omega^2}{\frac{7}{10}mr^2\omega^2} = \frac{2}{7}$$

Rolling on Inclined Surface

A body of mass M and radius R rolling down a plane inclined at an angle θ with the horizontal. The body rolls without slipping.



$$Ma = Mg\sin\theta - f$$

torque acting on the body $\tau = I\alpha = fR$

$$\Rightarrow \quad f = \frac{I\alpha}{R} = \frac{Ia}{R^2} \qquad (a = \alpha R)$$

so we can write $\quad Ma = Mg\sin\theta - \dfrac{Ia}{R^2} \Rightarrow a = g\sin\theta - \dfrac{Ia}{MR^2}$

$$\text{or} \quad a = \frac{g\sin\theta}{1 + \dfrac{I}{MR^2}} \quad \Rightarrow \quad a = \frac{g\sin\theta}{1 + \dfrac{K^2}{R^2}}$$

$$\Rightarrow \quad a = \frac{g\sin\theta}{1 + n^2} \qquad \left[I = MK^2 \text{ and let } \frac{K^2}{R^2} = n^2\right]$$

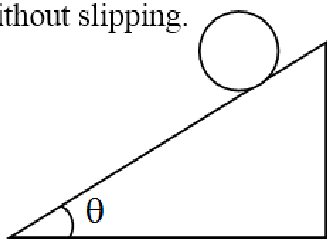
$$a = \frac{g\sin\theta}{1 + \dfrac{I_{cm}}{MR^2}}$$

frictional force f acting on the body : $f = Mg\sin\theta - M\dfrac{g\sin\theta}{1 + \dfrac{K^2}{R^2}} = Mg\sin\theta\left[\dfrac{\dfrac{K^2}{R^2}}{1 + \dfrac{K^2}{R^2}}\right] = Mg\sin\theta\left[\dfrac{n^2}{1+n^2}\right]$

$$f = \frac{Mg\sin\theta}{\dfrac{MR^2}{I_{cm}} + 1}$$

Example

A spherical shell of radius R is rolling down an incline of inclination θ without slipping. Find minimum value of coefficient of friction -



(A) $\frac{2}{7}\tan\theta$

(B) $\frac{2}{5}\tan\theta$

(C) $\frac{2}{3}\tan\theta$

(D) none

Sol.

Comparison between formula of translatory motion and rotatory motion

Translatory Motion	Rotatory Motion
$\vec{F} = \frac{d\vec{p}}{dt}$	$\vec{\tau} = \frac{d\vec{L}}{dt}$
$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$
◉ Linear momentum $(\vec{P})$	◉ Angular momentum $(\vec{L})$
$\vec{P} = m\vec{v}$	$\vec{J} = I\vec{\omega}$
◉ Linear kinetic energy	◉ Rotational kinetic energy
$KE = \frac{1}{2}mv^2$	$E = \frac{1}{2}I\omega^2$
◉ Work done $W = \vec{F}.\vec{S}$ (constant force)	◉ Work done $W = \vec{\tau}.\vec{\theta}$ (constant torque)
◉ Variable force	◉ Variable torque
$W = \int \vec{F}.d\vec{s}$	$W = \int \vec{\tau}.d\vec{\theta}$
◉ Power in linear motion	◉ Power in rotational motion
$P = \frac{dW}{dt} = \frac{\vec{F}.d\vec{s}}{dt}$	$P = \frac{dW}{dt} = \frac{\vec{\tau}.d\vec{\theta}}{dt}$
$P = \vec{F}.\vec{V}$	$P = \vec{\tau}.\vec{\omega}$
◉ Work energy theorem in T. M.	◉ Work energy theorem in R. M.
$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$	$W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$
◉ Linear impulse	◉ Angular impulse
It is product of large force for small time	It is product of large torque for small time
$\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$	$\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}$
$\Delta\vec{p} = \vec{F}\Delta t$ = Impulse	$\Delta\vec{L} = \vec{\tau}\Delta t$ = Angular impulse
Impulse momentum theorem	Angular Impulse momentum theorem

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