Binomial Theorem

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Concepts

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BINOMIAL THEOREM :

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **Binomial Theorem** .

If $x, y \in R$ and $n \in N$, then ;

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}y^2 + \ldots\ldots + {}^nC_r\, x^{n-r}y^r + \ldots\ldots + {}^nC_n y^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}\, y^r.$$

This theorem can be proved by Induction .

$$(x+y)^n = {}^nC_0\, x^{n-0} y^0 + {}^nC_1\, x^{n-1} y^1 \; - - \; \_\_\_ \quad {}^nC_n\, x^{n-n} y^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$$

$$T_{r+1} = {}^nC_r\, x^{n-r} y^r.$$

$(x+y)^5 = \_\_ + \_\_ + \_\_ + \_\_ + \_\_ + \_\_$

$= {}^5C_0 x^5 y^0 + {}^5C_1 x^4 y^1 + {}^5C_2 x^3 y^2 + {}^5C_3 x^2 y^3 + {}^5C_4 x^1 y^4 + {}^5C_5 x^0 y^5$

OBSERVATIONS :

(i) The number of terms in the expansion is $(n+1)$ i.e. one or more than the index .

(ii) The sum of the indices of x & y in each term is n .

(iii) The binomial coefficients of the terms ${}^nC_0, {}^nC_1$ .... **equidistant** from the beginning and the end are equal.

Problems

Expand : $(y + 2)^6$.

$^6C_0\, y^6 2^0 + {}^6C_1\, y^5 2^1 \ldots \ldots \; {}^6C_6\, y^0 2^6$

$\boxed{T_{r+1} = {}^6C_r\, y^{6-r}\, 2^r}$

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IMPORTANT TERMS IN THE BINOMIAL EXPANSION ARE:

(i) General term
(ii) Middle term
(iii) Term independent of x &
(iv) Numerically greatest term

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(i) The general term or the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is given by : $T_{r+1} = {}^nC_r\, x^{n-r} \cdot y^r$

(ii) The middle term(s) is the expansion of $(x+y)^n$ is (are) :

(a) If n is even, there is only one middle term which is given by ;

$$T_{(n+2)/2} = {}^nC_{n/2} \cdot x^{n/2} \cdot y^{n/2}$$

(b) If n is odd, there are two middle terms which are :

$$T_{(n+1)/2} \quad \& \quad T_{[(n+1)/2]+1}$$

(iii) Term independent of x contains no x ; Hence find the value of r for which the exponent of x is zero.

$(x+y)^n =$ $\boxed{T_{r+1} = {}^nC_r\, x^{n-r} y^r}$

→ No of terms = $\boxed{n+1}$

If n+1 is odd
1 Middle term

If (n+1) is even
2 Middle terms.

1, 3, 5, 7, 9

1, 3, 5, 7, 9, 11

odd terms → 1 Middle Term.

even terms → 2 Middle Terms

Total terms → 19, Middle term = $\left(\frac{19+1}{2}\right)^{th}$

Total terms → 20, Mid terms → 10^{th} & 11^{th}.

Problems

Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

$T_{r+1} = {}^{1000}C_r \, (9^{1/4})^{1000-r} \, (8^{1/6})^r$ $r = 0, 1, 2 \ldots \ldots 1000$

$= {}^{1000}C_r \times 3^{\frac{1000-r}{2}} \times 2^{r/2}$

$= {}^{1000}C_r \, 3^{500 - r/2} \times 2^{r/2}$

This term will be rational when $r/2$ is integer.

thus $r = 0, 2, 4, 6 \ldots \ldots 1000$

No of rational terms $= 501$

Problems

The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is -

(A) 1 (B) $\frac{5}{12}$ (C) ${}^{10}C_1$ (D) none of these ✓

$$T_{r+1} = {}^{10}C_r \left(\sqrt{\frac{x}{3}}\right)^{10-r} \times \left(\sqrt{\frac{3}{2x^2}}\right)^{r}$$

$$= {}^{10}C_r \times \left(\frac{1}{\sqrt{3}}\right)^{10-r} \times (\sqrt{x})^{10-r} \times (\sqrt{3})^{r} \times \frac{1}{(\sqrt{2x^2})^{r}}$$

$$= {}^{10}C_r \times \frac{1}{3^{\frac{10-r}{2}}} \times \frac{1}{2^{r/2}} \times 3^{r/2} \times \frac{x^{(10-r)/2}}{x^{r}}$$

power of $x = 0$

$$\frac{10-r}{2} - r = 0$$

$$\frac{10-r-2r}{2} = 0$$

$$10-3r = 0$$

$$r = 10/3 \quad \text{N.P.}$$

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(iv) To find the Numerically greatest term is the expansion of $(1 + x)^n$, $n \in N$ find $\boxed{\frac{T_{r+1}}{T_r} = \frac{{}^nC_r x^r}{{}^nC_{r-1} x^{r-1}} = \frac{n-r+1}{r} x}$. Put the absolute value of x & find the value of r Consistent with the inequality $\frac{T_{r+1}}{T_r} > 1$.

$$\boxed{\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times x}$$

Problems

$(3-5x)^{11} = {}^{11}C_0\, 3^{11}(-5x)^0 + {}^{11}C_1\, x\, 3^{10}(-5x)^1$ ———

Find numerically greatest term in the expansion of $(3 - 5x)^{11}$ when $x = \frac{1}{5}$

T_2 & T_3 are numerically greatest.

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \times x$$

$$= \frac{11-r+1}{r} \times \frac{1}{5} > 1$$

$$\frac{12-r}{5r} > 1$$

$$12 - r > 5r$$

$$12 > 6r \Rightarrow 6r < 12 \Rightarrow r < 2$$

When $r < 2$ $\quad \frac{T_{r+1}}{T_r} > 1 \Rightarrow T_{r+1} > T_r$

$r < 2$	$T_{r+1} > T_r$
$r = 2$	$T_{r+1} = T_r$
$r > 2$	$T_{r+1} < T_r$.

$T_1 < T_2 = T_3 > T_4 > T_5 > T_6 > T_7 > T_8 > T_9 > T_{10} >$

$T_{11} > T_{12}$

BINOMIAL COEFFICIENTS & THEIR PROPERTIES :

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Propertices of ^nC_r

$^nC_r = {}^nC_{n-r}$ $\quad \boxed{^{10}C_3 = {}^{10}C_7}$ $\quad {}^nC_x = {}^nC_{n-y}$

$x = n - y \Rightarrow x + y = n$

(1) $^nC_r = {}^nC_{n-r} \Rightarrow {}^nC_x = {}^nC_y$ has two solution $x = y$ or $x + y = n$.

(2) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(3) $^nC_r = \frac{n}{r}\left({}^{n-1}C_{r-1}\right)$

(4) $\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}$.

$(x+y)^n = {}^nC_0 x^{n-0} y^0 + {}^nC_1 x^{n-1} y^1 - - - {}^nC_n x^{n-n} y^n$

$\hookrightarrow$ Identity

$(x+y)^2 = x^2 + 2xy + y^2$

2) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$^{10}C_6 + {}^{10}C_5 = {}^{11}C_6$.

$(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 - - \quad {}^nC_n x^n$

$(1-x)^n = {}^nC_0 x^0 - {}^nC_1 x^1 + {}^nC_2 x^2 \quad \_\_ \quad \_\_\_$

3) $^nC_r = \frac{n}{r} \times {}^{n-1}C_{r-1}$

4) $\boxed{\frac{^nC_r}{^nC_{r-1}} = \frac{n-r+1}{r}}$

Problems

$^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

$^{25}C_{10} + {}^{24}C_{10} + \dots\dots\dots + {}^{10}C_{10} = {}^{n}C_{r}$

$^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + {}^{13}C_{10} \;- - - - - - \; {}^{24}C_{10} + {}^{25}C_{10}$

$^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + {}^{13}C_{10} \;- - - - \; {}^{24}C_{10} + {}^{25}C_{10}$

$^{12}C_{11} \quad {}^{13}C_{11} \quad {}^{14}C_{11}$

$^{25}C_{11} + {}^{25}C_{10} = {}^{26}C_{11}$.

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots\ldots + {}^nC_r x^r + \ldots\ldots + {}^nC_n x^n$ ...(1)

(a) **Sum of Coefficient :** putting x = 1 in (1), we get

$C_0 + C_1 + C_2 + \ldots\ldots + C_n = 2^n$ — 1 ...(2)

(b) **Sum of coefficients with alternate signs :** putting x = –1 in(1)

We get

$C_0 - C_1 + C_2 - C_3 + \ldots\ldots = 0$ — 2 ...(3)

(c) **Sum of coefficients of even and odd terms:** from (3), we have

$C_0 + C_2 + C_4 + \ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots$ ...(4)

i.e. sum of coefficients of even and odd terms are equal.

from (2) and (4)

$\Rightarrow \quad C_0 + C_2 + \ldots\ldots = C_1 + C_3 + \ldots\ldots = 2^{n-1}$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 - - {}^nC_n x^n$

${}^nC_0 + {}^nC_1 + {}^nC_2 - - {}^nC_n$ //

$x = 1 \quad 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 - - {}^nC_n$

$2(C_0 + C_2 + C_4 - -) = 2^n$

$C_0 + C_2 + C_4 - - = 2^{n-1}$

(d) Sum of products of coefficients : Replacing x by 1/x in (1)

We get

$$\left(1+\frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots + \frac{C_n}{x^n} + \ldots$$

$\boxed{{}^nC_0 \times {}^nC_0 + {}^nC_1 \times {}^nC_1 + {}^nC_2 \times {}^nC_2 \,--\, = {}^{2n}C_n}$

${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 \,--\, {}^nC_n^2 = {}^{2n}C_n$

Multiplying (1) by (5), we get

$$\frac{(1+x)^{2n}}{x^n} = (C_0 + C_1x + C_2x^2 + \ldots)\left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \ldots\right)$$

Now, comparing coefficients of x^r on both the sides, we get

$$C_0C_r + C_1C_{r+1} + \ldots\ldots + C_{n-r}C_n = {}^{2n}C_{n-r}$$

$$= \frac{2n!}{(n+r)!(n-r)!}$$

** $\boxed{{}^nC_0 \times {}^nC_r + {}^nC_1 \times {}^nC_{r+1} + {}^nC_2 \times {}^nC_{r+2} \,--\, {}^nC_{n-r} \times {}^nC_r = {}^{2n}C_{n-r}}$

${}^nC_0 \times {}^nC_2 + {}^nC_1 \times {}^nC_3 + {}^nC_2 \times {}^nC_4 \,--\, = {}^{2n}C_{n-2}$

${}^nC_0 \times {}^nC_1 + {}^nC_1 \times {}^nC_2 + {}^nC_2 \times {}^nC_3 \,--\, = {}^{2n}C_{n-1}$

(e) **Sum of squares of coefficients :**

putting r = 0 in (6), we get

$$C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \frac{2n!}{n!\,n!}$$

$${}^{2n}C_n = \frac{2n!}{n!\,n!}$$

(f) putting r = 1 in (6), we get

$$C_0\,C_1 + C_1C_2 + C_2C_3 + \ldots\ldots. + C_{n-1}\,C_n = {}^{2n}C_{\,n-1}$$

$$= \frac{2n!}{(n+1)!\,(n-1)!}$$

(g) putting r = 2 in (6) , we get

$$C_0C_2 + C_1C_3 + C_2C_4 + \ldots. + C_{n-2}C_n = {}^{2n}C_{n-2}$$

$$= \frac{2n!}{(n+2)!\,(n-2)!}$$

$(x+y)^n$ n is a natural no

BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...\infty$ Provided $\boxed{|x| < 1.}$

Note:

(i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. $(n+1)$ & the coefficient of successive terms are: ${}^nC_0, {}^nC_1, {}^nC_2, {}^nC_3 ..... {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the Coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - .... \infty$ **(b)** $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + ... \infty$

(c) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ... \infty$ **(d)** $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ... \infty$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 \ldots \infty$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 \ldots \infty$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 \ldots \infty$

A.G.P.