Binomial Theorem

Problem Solving (JEE Mains)

Vishal Garg (B.Tech, IIT Bombay)

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JEE Mains Problems

The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1+x+x^2)^9$ is (2019 Main, 12 April I)

(a) 84 (b) -126
(c) -84 (d) 126

$1^3 - x^3 = (1-x)(1+x^2+x)$

${}^nC_r = {}^nC_{n-r}$

$(1+x)(1-x)(1-x)^9(1+x+x^2)^9$

$= 1 (1-x^2)[(1-x)(1+x+x^2)]^9$

$= (1-x^2)(1-x^3)^9$

$= 1 \times (1-x^3)^9 - x^2 \times (1-x^3)^9$

$= 1 \times \text{coeff of } x^{18} - x^2 \times \text{coeff of } x^{16}$

$= 1 \times {}^9C_6 - x^2 \times 0 = {}^9C_6 = {}^9C_3 = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$

$(1-x^3)^9 = \_\_ \_\_\_\_$

$T_{r+1} = {}^9C_r (-x^3)^r$

$= {}^9C_r (-1)^r x^{3r}$

coeff of x^{18}: $3r = 18 \Rightarrow r = 6$

$= {}^9C_6 (-1)^6$

$= {}^9C_6$

coeff of x^{16}: $3r = 16 \Rightarrow r = 16/3$ ✗

$= 0$

JEE Mains Problems

If the coefficients of x^2 and x^3 are both zero, in the expansion of the expression $(1 + ax + bx^2)(1 - 3x)^{15}$ in powers of x, then the ordered pair (a, b) is equal to

(2019 Main, 10 April I)

(a) (28, 315) ✓ (b) (– 21, 714)

(c) (28, 861) (d) (– 54, 315)

$(1-3x)^{15} \equiv T_{r+1} = {}^{15}C_r(-3x)^r$

coeff of

$x^2 = {}^{15}C_2 \times (-3)^2 = \frac{15\times 14}{2}\times 9 = 945$

$x^3 = {}^{15}C_3 \times (-3)^3 = \frac{15\times 14\times 13}{3\times 2}\times 27 = -455\times 27$

$x = {}^{15}C_1 \times (-3) = -45$

$x^0 = {}^{15}C_0 \times 1 = 1$

$\rightarrow (1+ax+bx^2)(1-3x)^{15}$

$1\times(1-3x)^{15} + ax\times(1-3x)^{15} + bx^2\times(1-3x)^{15}$

overall coeff of x^2 = coeff of x^2 in $(1-3x)^{15}$ + a × coeff of x^1 in $(1-3x)^{15}$ + b × coeff of x^0

$= 945 + a(-45) + b\times 1 = 0$

$-45a + b = -945 \Rightarrow \boxed{45a - b = 945}$ — (1)

overall coeff of x^3 = coeff of x^3 + a × coeff of x^2 + b × coeff of x

$= -455\times 27 + a\times 945 - 45b = 0 \Rightarrow \boxed{945a - 45b = 455\times 27}$ — (2)

JEE Mains Problems

The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\cdot\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to **(2019 Main, 12 April II)**

(a) – 72 (b) 36 (c) – 36 (d) – 108

$$\frac{1}{60}\times\left(2x^2-\frac{3}{x^2}\right)^6 - \frac{x^8}{81}\left(2x^2-\frac{3}{x^2}\right)^6$$

Coeff of x^0

Coeff of x^{-8}

$$\frac{1}{\cancel{60}_3}\times \cancel{20}\times 8\,(-\cancel{27}^{9}) - \frac{1}{81}\times 6\times 2\times(-3)^5$$

$$-72+\cancel{54}\;36$$

$$\left(2x^2-\frac{3}{x^2}\right)^6$$

$$T_{r+1} = {}^6C_r\,(2x^2)^{6-r}\left(-\frac{3}{x^2}\right)^r$$

$$= {}^6C_r\times 2^{6-r}(-3)^r\,x^{12-2r-2r}$$

$$T_{r+1} = {}^6C_r\,2^{6-r}(-3)^r\,x^{12-4r}$$

For $x^0 \equiv 12-4r=0$, $r=3$

coeff of $x^0 = {}^6C_3\,2^{6-3}(-3)^3$

$$= \frac{6\times5\times4}{3\times2}\times 2^3(-3)^3$$

$$= 20\times 8\times(-27)$$

For x^{-8}

$12-4r=-8 \Rightarrow 4r=20$, $r=5$

coeff $= {}^6C_5\,2^1(-3)^5$

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JEE Mains Problems

$^nC_x = {^nC_y}$
$\to x=y$
$\to x+y=n$

The smallest natural number n, such that the coefficient of x in the expansion of $\left(x^2+\dfrac{1}{x^3}\right)^n$ is $^nC_{23}$, is

(2019 Main, 10 April II)

(a) 35 (b) 23 (c) 58 (d) 38

$\left(x^2+\frac{1}{x^3}\right)^n$

$T_{r+1} = {^nC_r}(x^2)^{n-r}\left(\frac{1}{x^3}\right)^r$

$= {^nC_r}\, x^{2n-2r-3r}$

$= {^nC_r}\, x^{2n-5r}$

For $x^1 \equiv 2n-5r=1 \Rightarrow 5r=2n-1$

$\boxed{r=\frac{2n-1}{5}}$

$2n=5r+1$
$n=(5r+1)/2$

$^nC_r = {^nC_{23}}$

Case-1 $r=23$
$n=\frac{5r+1}{2}$
$=\frac{116}{2}=58$

Case-2
$r+23=n$
$r=n-23$
$2n=5(n-23)+1$
$2n=5n-115+1$
$3n=114$
$n=38$

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JEE Mains Problems

If some three consecutive coefficients in the binomial expansion of $(x+1)^n$ in powers of x are in the ratio $2:15:70$, then the average of these three coefficients is

(2019 Main, 9 April II)

(a) 964 (b) 227 (c) 232 (d) 625

$$(x+1)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} - \text{—}$$

$$\boxed{{}^nC_r : {}^nC_{r+1} : {}^nC_{r+2} = 2:15:70}$$

$$\frac{{}^nC_r + {}^nC_{r+1} + {}^nC_{r+2}}{3} = \frac{{}^{16}C_1 + {}^{16}C_2 + {}^{16}C_3}{3} = \frac{16 + \frac{16\times 15}{2} + \frac{16\times 15\times 14}{3\times 2}}{3}$$

$$= \frac{16+120+560}{3}$$

232
696
3

$$\frac{15}{2} = \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{\frac{n!}{(r+1)!\,(n-r-1)!}}{\frac{n!}{r!\,(n-r)!}} = \frac{n-r}{r+1}$$

$$\frac{14}{3} = \frac{70}{15} = \frac{{}^nC_{r+2}}{{}^nC_{r+1}} = \frac{\frac{n!}{(r+2)!\,(n-r-2)!}}{\frac{n!}{(r+1)!\,(n-r-1)!}} = \frac{n-r-1}{r+2}$$

$$\frac{15}{2} = \frac{n-r}{r+1} \Rightarrow 15r+15 = 2n-2r$$

$$\boxed{17r = 2n-15}$$

$$\frac{14}{3} = \frac{n-r-1}{r+2} \Rightarrow 14r+28 = 3n-3r-3$$

$$17r = 3n-31$$

$$3n-31 = 2n-15$$

$$\boxed{n=16,\ r=1}$$

JEE Mains Problems

If the fourth term in the binomial expansion of $\left(\frac{2}{x}+x^{\log_8 x}\right)^6$ $(x>0)$ is 20×8^7, then the value of x is (2019 Main, 9 April I)

(a) 8^{-2} (b) 8^3

(c) 8 (d) 8^2

$T_{r+1} = {}^6C_r\left(\frac{2}{x}\right)^{6-r}\left(x^{\log_8 x}\right)^r$

For 4th term, $r=3$

$T_4 = {}^6C_3\left(\frac{2}{x}\right)^3\left(x^{\log_8 x}\right)^3 = 20\times 8^7$

$\Rightarrow \frac{6\times5\times4}{3\times2}\times\frac{8}{x^3}\times\left(x^{\log_8 x}\right)^3 = 20\times 8^7$

${}^6C_3 = \frac{6!}{3!\,3!} = \frac{6\times5\times4\times 3!}{3!\times 3!} = \frac{6\times5\times4}{3\times2\times1}$

$\frac{\left(x^{\log_8 x}\right)^3}{x^3} = 8^6 \Rightarrow \frac{x^{\log_8 x}}{x} = 8^2$

$x^{\log_8 x} = 64x$

$\log_8 x^{\log_8 x} = \log_8 64x$

$(\log_8 x)^2 = \log_8 64 + \log_8 x$

$\boxed{(\log_8 x)^2 - \log_8 x - 2 = 0}$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$y = 2,\ y = -1$

$\log_8 x = 2,\ \log_8 x = -1$

$x = 8^2,\ x = 8^{-1}$

$\log_{10} x = -4$ $\quad \log_{10} x = 1$

$x = 10^{-4}$, $x = 10^{1}$

$1 + \log_{10} x = \log_{10} 10 + \log_{10} x = \log_{10} 10x$

JEE Mains Problems

If the fourth term in the binomial expansion of $\left(\sqrt{x^{\left(\frac{1}{1+\log_{10} x}\right)}} + x^{\frac{1}{12}}\right)^6$ is equal to 200, and $x > 1$, then the value of x is **(2019 Main, 8 April II)**

(a) 100 (b) 10^4

(c) 10 (d) 10^3

$\sqrt{x^{\frac{1}{1+\log_{10} x}}} = \sqrt{x^{\frac{1}{\log_{10} 10x}}}$

$y = \sqrt{x^{\log_{10x} 10}}$

$T_{3+1} = {}^6C_3 \, y^3 \times (x^{1/12})^3$

$= {}^6C_3 \, y^3 \times x^{1/4} = 200$

$\Rightarrow \cancel{6 \times 5 \times 4} \times y^3 \times x^{1/4} = \cancel{200}\, 10$

$\cancel{3 \times 2 \times 1}$

$y^3 \times x^{1/4} = 10$

$y^{12} \times x^{1} = 10^4$

$\boxed{\left(\sqrt{x^{\log_{10x} 10}}\right)^{12} \times x = 10^4}$

$\left(x^{\log_{10x} 10}\right)^6 \times x = 10^4$

$x^{6\log_{10x} 10} = \frac{10^4}{x}$

$6 \log_{10x} 10 \log_{10} x = 4 - \log_{10} x$

$\frac{6 \log_{10} x}{1 + \log_{10} x} = 4 - \log_{10} x$

$\frac{6z}{1+z} = 4 - z$

$6z = 4 - z + 4z - z^2$

$z^2 + 3z - 4 = 0 \Rightarrow (z+4)(z-1) = 0$

$z = -4, 1$

JEE Mains Problems

The sum of the coefficients of all even degree terms is x in the expansion of (2019 Main, 8 April I)

$(x+\sqrt{x^3-1})^6+(x-\sqrt{x^3-1})^6,\ (x>1)$ is equal to

(a) 29 (b) 32 (c) 26 (d) 24

$\sqrt{x^3-1}=y$

$(x+y)^6+(x-y)^6$

$[{}^6C_0x^6y^0+\cancel{{}^6C_1x^5y^1}+{}^6C_2x^4y^2--{}^6C_6x^0y^6]+({}^6C_0x^6y^0-\cancel{{}^6C_1}x^5y^1+{}^6C_2x^4y^2--\ ]$

$2[{}^6C_0x^6+{}^6C_2x^4y^2+{}^6C_4x^2y^4+{}^6C_6y^6]$

$2[1\times x^6+\frac{6\times5}{2\times1}\times x^4(x^3-1)+\frac{6\times5}{2\times1}\times x^2(x^3-1)^2+1(x^3-1)^3]$

$2[x^6+15(x^7-x^4)+15(x^2(x^6+1-2x^3)+x^9-1-3x^3(x^3-1)$

$2[1-15+15+15-1-3]=12\times2=24$

JEE Mains Problems

The total number of irrational terms in the binomial expansion of $(7^{1/5} - 3^{1/10})^{60}$ is (2019 Main, 12 Jan II)

(a) 49 (b) 48 (c) 54 (d) 55

$T_{r+1} = {}^{60}C_r (7^{1/5})^{60-r} (-3^{1/10})^r$

$= {}^{60}C_r \, 7^{12-r/5} (-3)^{r/10}$

For Rational terms r should be multiple of 10

$r = 0, 10, 20, 30, 40, 50, 60$

For 7 values of r, terms are rational

Total terms $= 61$ Rational $= 7$

Irrational $= 61 - 7 = 54$.

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JEE Mains Problems

The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$ equals 5670 is (2019 Main, 11 Jan I)

(a) 4 (b) 0 (c) 6 (d) 8

Total terms = 9
Middle Term will be 5^{th} term.

$T_5 = 5670$

$^8C_4\left(\frac{x^3}{3}\right)^4 \times \left(\frac{3}{x}\right)^4 = 5670$

$\frac{8\times7\times6\times5}{4\times3\times2\times1} \times \frac{x^{12}}{3^4} \times \frac{3^4}{x^4} = 5670$

$70\, x^8 = 5670$

$x^8 = 81$

$x^8 = 81$

$x^8 = 3^4$

$x^2 = 3$

$x = \pm\sqrt{3}$

$\sqrt{3} + (-\sqrt{3}) = 0$

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JEE Mains Problems

The positive value of λ for which the coefficient of x^2 in the expression $x^2\left(\sqrt{x}+\frac{\lambda}{x^2}\right)^{10}$ is 720, is

(2019 Main, 10 Jan II)

(a) 3 (b) $\sqrt{5}$ (c) $2\sqrt{2}$ (d) 4

coeff of x^0

$\left(\sqrt{x}+\frac{\lambda}{x^2}\right)^{10}$ coeff of x^0

$T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r} \left(\frac{\lambda}{x^2}\right)^r$

$= {}^{10}C_r \, x^{5-\frac{r}{2}} \frac{\lambda^r}{x^{2r}} = {}^{10}C_r \lambda^r \, x^{5-\frac{r}{2}-2r}$

$5-\frac{r}{2}-2r=0$

$5-\frac{5r}{2}=0 \Rightarrow r=2$

coeff is ${}^{10}C_2 \times \lambda^2 = 720$

$\frac{\cancel{10}\times \cancel{9}}{2} \times \lambda^2 = \cancel{720}\ 8$

$\lambda^2 = 16$

$\lambda = \pm 4$

JEE Mains Problems

If the third term in the binomial expansion of $(1 + x^{\log_2 x})^5$ equals 2560, then a possible value of x is

(2019 Main, 10 Jan I)

(a) $4\sqrt{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $2\sqrt{2}$

$$^5C_2 \times (x^{\log_2 x})^2 = 2560$$

$$\cancel{\frac{5\times 4}{2}} \times (x^{\log_2 x})^2 = 256\cancel{0}$$

$$(x^{\log_2 x})^2 = 256$$

$$x^{\log_2 x} = \pm 16$$

$$(\log_2 x)(\log_2 x) = \log_2 16$$

$$(\log_2 x)^2 = 4$$

$$\log_2 x = \pm 2$$

$$x = 2^2, \ 2^{-2}$$

$$4, \ \frac{1}{4}$$

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JEE Mains Problems

The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is

(2019 Main, 9 Jan II)

(a) 12 (b) 10 (c) 15 (d) 14

$$\frac{1-t^6}{1-t} = \frac{(1-t^2)(1+t^4+t^2)}{1-t} = \left[(1+t)(1+t^2+t^4)\right]^3$$

$$= (1+t)^3(1+t^2+t^4)^3$$

$$= (1+t^3+3t^2+3t)(1+\underbrace{t^2+t^4})^3$$

$$= (1+t^3+3t^2+3t)\left[1+(t^2+t^4)^3+3(t^2+t^4)^2+3(t^2+t^4)\right]$$

$$(1+3t+3t^2+t^3)(1+t^6\cancel{(1+t^2)^3}+3t^4(1+t^2)^2+3t^2+3t^4)$$

$$(1+3t+3t^2+t^3)(1+3t^4+3t^2+3t^4) = (1+3t+3t^2+t^3)(1+3t^2+6t^4)$$

$$6+9 = 15$$

JEE Mains Problems

The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \ldots + ({}^{21}C_{10} - {}^{10}C_{10})$ is **(2017 Main)**

(a) $2^{21} - 2^{11}$ (b) $2^{21} - 2^{10}$

(c) $2^{20} - 2^{9}$ ✗ (d) $2^{20} - 2^{10}$ ✓

${}^nC_0 + {}^nC_1 + {}^nC_2 - - {}^nC_n = 2^n$

${}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 - - {}^{10}C_{10} = 2^{10}$

${}^{10}C_1 + {}^{10}C_2 - - {}^{10}C_{10} = 2^{10} - 1$

$[\underbrace{{}^{21}C_1 + {}^{21}C_2 + {}^{21}C_3 - - {}^{21}C_{10}}] - [\underbrace{{}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 - - {}^{10}C_{10}}]$

$2^{20} - 1 - (2^{10} - 1) = 2^{20} - 2^{10}$

${}^{21}C_0 + \underbrace{{}^{21}C_1 + {}^{21}C_2 - - {}^{21}C_9 + {}^{21}C_{10}}_{S} + \underbrace{{}^{21}C_{11} - - {}^{21}C_{20}}_{S} + {}^{21}C_{21} = 2^{21}$

$1 + S + S + 1 = 2^{21}$

$1 + S + S + 1 = 2^{21} \Rightarrow 2S = 2^{21} - 2 \Rightarrow S = 2^{20} - 1$

JEE Mains Problems

Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1 x + a_2 x^2 + \ldots + a_{50} x^{50}$, for all $x \in R$; then $\frac{a_2}{a_0}$ is equal to (2019 Main, 11 Jan II)

(a) 12.25 (b) 12.50 (c) 12.00 (d) 12.75

$${}^{50}C_0 x^{50} 10^0 + {}^{50}C_1 x^{49} \times 10^1 + {}^{50}C_2 x^{48} 10^2 \ldots - {}^{50}C_{50} x^0 10^{50} + (\quad)$$

$$2\left({}^{50}C_0 x^{50} + {}^{50}C_2 x^{48} 10^2 + {}^{50}C_4 x^{46} \times 10^4 \ldots - {}^{50}C_{48} x^2 10^{48} + {}^{50}C_{50} x^0 \times 10^{50}\right)$$

$$\frac{a_2 = 2 \times {}^{50}C_{48} \times 10^{48}}{a_0 = 2 \times {}^{50}C_{50} \times 10^{50}} = \frac{{}^{50}C_{48}}{10^2} = \frac{\cancel{50} \times 49}{2 \times 1 \times \cancel{100}_2} = \frac{49}{4} = 12.25$$

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JEE Mains Problems

$^nC_r = {^nC_{n-r}}$

$^nC_0 \times {^nC_r} + {^nC_1} \times {^nC_{r+1}} \ldots {^nC_{n-r}} \times {^nC_n}$

$= {^{2n}C_{n-r}}$

The value of r for which

$^{20}C_r \, ^{20}C_0 + {^{20}C_{r-1}} \, ^{20}C_1 + {^{20}C_{r-2}} \, ^{20}C_2 + \ldots + {^{20}C_0} \, ^{20}C_r$

is maximum, is (2019 Main, 11 Jan I)

(a) 15 (b) 10 (c) 11 (d) 20

$^{20}C_{20-r} \times {^{20}C_0} + {^{20}C_{21-r}} \times {^{20}C_1} + {^{20}C_{22-r}} \times {^{20}C_2} - - {^{20}C_{20}} \times {^{20}C_r}$

$^{20}C_0 \times {^{20}C_{20-r}} + {^{20}C_1} \times {^{20}C_{21-r}} + {^{20}C_2} \times {^{20}C_{22-r}} - - {^{20}C_r} \times {^{20}C_{20}}$

$= {^{2\times 20}C_{20-(20-r)}} = {^{40}C_r} \rightarrow$ Maximum

So $r = 20$

$^{40}C_{20}$
