Differentiation

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First Principle of Derivative

The derivative can be defined for $f\ (x)$ at any point x as

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right].$$

If the function is given as $y = f\ (x)$ then its derivative is written as $\frac{dy}{dx} = f'(x).$

STANDARD DERIVATIVES :

(i) $\frac{d}{dx} x^n = nx^{n-1}, x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \ln a$

(iv) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

Theorem on Derivatives

T-1 : $\frac{d}{dx}(f_1(x) \pm f_2(x)) = \frac{d}{dx} f_1(x) \pm \frac{d}{dx} f_2(x)$.

T-2 : $\frac{d}{dx}(kf(x)) = k\frac{d}{dx} f(x)$, where k is any constant.

T-3 : PRODUCT RULE :

$$\frac{d}{dx}\{f_1(x)\, f_2(x)\} = f_1(x)\frac{d}{dx} f_2(x) + f_2(x)\frac{d}{dx} f_1(x).$$

Note : If 3 functions are involved then remember

$$D\left(f(x)\cdot g(x)\cdot h(x)\right) = f(x)\cdot g(x)\cdot h'(x) + g(x)\cdot h(x)\cdot f'(x) + h(x)\cdot f(x)\cdot g'(x)$$

Theorem on Derivatives

T-4 QUOTIENT RULE :

$$y = \frac{f(x)}{g(x)}$$

$$D\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{g^2(x)}$$, to be remembered as

$$D\left(\frac{N^r}{D^r}\right) = \frac{D^r \frac{d}{dx}(N^r) - N^r \frac{d}{dx}(D^r)}{(D^r)^2};$$

Theorem on Derivatives

T-5 DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE)

If $y = f(u)$ & $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ " CHAIN RULE "

The derivative of a composite function can also be expressed as follows. $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is defined then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

Theorem on Derivatives

T-5 DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE)

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n\,[f(x)]^{n-1} \cdot f'(x)$
$\sqrt{f(x)}$	$\frac{f'(x)}{2\sqrt{f(x)}}$
$\frac{1}{[f(x)]^n}$	$-\frac{n \cdot f'(x)}{[f(x)]^{n+1}}$
$\sin\,[f(x)]$	$\cos\,[f(x)] \cdot f'(x)$
$\cos\,[f(x)]$	$-\sin\,[f(x)] \cdot f'(x)$
$\tan\,[f(x)]$	$\sec^2\,[f(x)] \cdot f'(x)$
$\sec\,[f(x)]$	$\sec\,[f(x)] \cdot \tan\,[f(x)] \cdot f'(x)$

y	$\frac{dy}{dx}$
$\cot\,[f(x)]$	$-\operatorname{cosec}^2\,[f(x)] \cdot f'(x)$
$\operatorname{cosec}\,[f(x)]$	$-\operatorname{cosec}\,[f(x)] \cdot \cot\,[f(x)] \cdot f'(x)$
$a^{f(x)}$	$a^{f(x)} \cdot \log a \cdot f'(x)$
$e^{f(x)}$	$e^{f(x)} \cdot f'(x)$
$\log\,[f(x)]$	$\frac{f'(x)}{f(x)}$
$\log_a\,[f(x)]$	$\frac{f'(x)}{f(x)\log a}$

Theorem on Derivatives

T-6 LOGARITHMIC DIFFERENTIATION :

To find the derivative of :

(i) a function which is the product or quotient of a number of functions

$y = f_1(x)\, f_2(x)\, f_3(x)\, ...$ or $y = \dfrac{f_1(x)\, f_2(x)\, f_3(x).....}{g_1(x)\, g_2(x)\, g_3(x).....}$.

(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convenient to take the logarithm of the function first & then differentiate

OR

$y = (f(x))^{g(x)} = e^{g(x).\ln(f(x))}$ and then differentiate.

Theorem on Derivatives

T-7 IMPLICIT FUNCTIONS :

If the variable x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible to express y as a function of x in the form $y = \phi(x)$, then such functions are said to be implicit functions.

DERIVATIVE OF IMPLICIT FUNCTION :

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit functions, both x & y are present.

A DIRECT FORMULA FOR IMPLICIT FUNCTIONS :

Let $f(x, y) = 0$. Take all the terms of left side and put left side equal to $f(x, y)$.

$$\text{Then } \frac{dy}{dx} = -\frac{\text{diff. of } f \text{ w.r.t. } x \text{ keeping } y \text{ as constant}}{\text{diff. of } f \text{ w.r.t. } y \text{ keeping } x \text{ as constant}}$$

Derivatives of Parametric Functions

If $x = f(t)$ and $y = g(t)$ are differentiable functions of t then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

DERIVATIVES OF ORDER TWO & THREE

the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Similarly, the 3rd order derivative of y w. r. t. x, if it exists, is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ It is also denoted by $f'''(x)$ or y'''.

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

(i) $$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{(1-x^2)}}$$

(ii) $$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{(1-x^2)}}$$

(iii) $$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

(iv) $$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

(v) $$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$$

(vi) $$\frac{d}{dx}(\mathrm{cosec}^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Differentiation by using Trigonometric Transformations :

With the help of trigonometric transformations, the labour involved in computing derivative can be reduced. Note some standard trigonometric substitutions

1. $\sqrt{a^2 - x^2} \Rightarrow$ put $x = a\sin\theta$ or $a\cos\theta$
2. $\sqrt{a^2 + x^2} \Rightarrow$ put $x = a\tan\theta$ or $a\cot\theta$
3. $\sqrt{x^2 - a^2} \Rightarrow$ put $x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
4. $\sqrt{(a - x)/(a + x)} \Rightarrow$ put $x = a\cos\theta$ or $a\cos 2\theta$
5. $\dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}} \Rightarrow$ put $x = a\sin 2\theta$

DERIVATIVE OF FUNCTIONS EXPRESSED IN THE DETERMINANT FORM :

Let $F(x) = \begin{vmatrix} f & g & h \\ u & v & w \\ l & m & n \end{vmatrix}$ where all functions are differentiable then

$$D'(x) = \begin{vmatrix} f' & g' & h' \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u' & v' & w' \\ l & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u & v & w \\ l' & m' & n' \end{vmatrix}$$

This result may be proved by first principle and the same operation can also be done column wise.

End of Lecture