Continuity & Differentiability

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By

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FORMULATIVE DEFINITION OF CONTINUITY

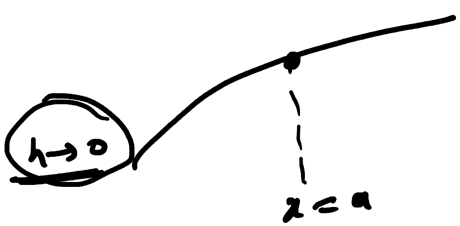
A function f(x) is said to be continuous at x = a, if $\lim_{x\to a} f(x)$ exists and = f (a).

Symbolically f is continuous at x = a if $\lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+h) = f(a)$ = a finite quantity.

L.H.L = R.H.L = f(a) = finite

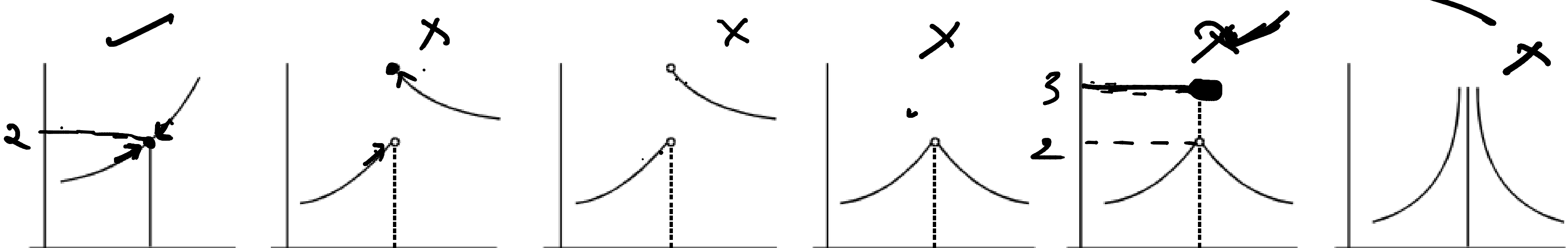
i.e. LHL at x = a = RHL at x = a = value of f (x) at x = a = a finite quantity.

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$



discontinous

continous



x = a figure (1) | x = a figure (2) | x = a figure (3) | x = a figure (4) | x = a figure (5) | x = a figure (6)

CONTINUITY IN AN INTERVAL

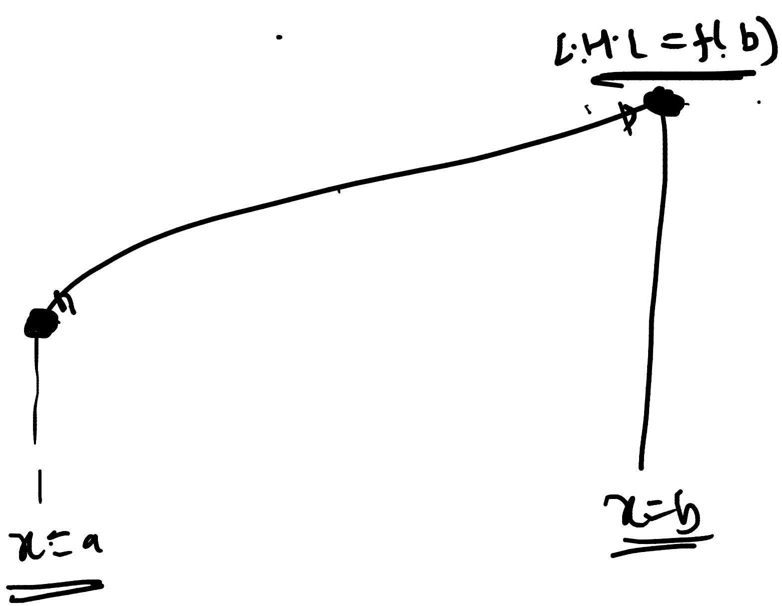
A function f is said to be continuous in (a, b) if f is continuous at each & every point $\in$(a, b).

A function f is said to be continuous in a closed interval $[a, b]$ if :

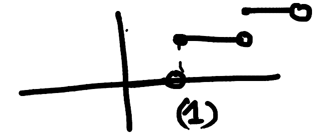
(i) f is continuous in the open interval (a , b) &

(ii) f is right continuous at **'a'** i.e. $\lim\limits_{x \to a^+} f(x) = f(a) =$ a finite quantity .

(iii) f is left continuous at **'b'** i.e. $\lim\limits_{x \to b^-} f(x) = f(b) =$ a finite quantity .



REASONS OF DISCONTINUITY : A function can be discontinuous due to the following reasons.



(i) $\lim\limits_{x\to a} f(x)$ does not exist ($f(a)$ may or may not be defined)

i.e. $\lim\limits_{h\to 0} f(a+h) \neq \lim\limits_{h\to 0} f(a-h)$

R·H·L ≠ L·H·L

e.g. $f(x) = [x]$; $f(x) = \text{sgn } x$

$f(x) = \dfrac{x}{x-1}$

(ii) $\lim\limits_{x\to a} f(x)$ exist but is not equal to $f(a)$

i.e. $\lim\limits_{h\to 0} f(a+h) = \lim\limits_{h\to 0} f(a-h) \neq f(a)$

$$f(x) = \begin{cases} (1-x)\tan\dfrac{\pi x}{2} & \text{if } x \neq 1 \\ \left(\dfrac{\pi}{2}\right) & \text{if } x = 1 \end{cases}$$

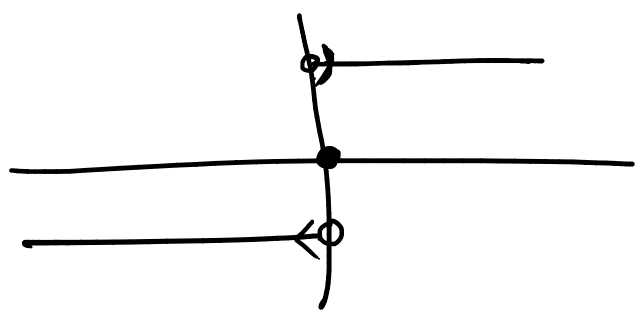
$\dfrac{1-x}{\cot\frac{\pi x}{2}}$

$\dfrac{-1}{-\operatorname{cosec}^2\frac{\pi x}{2}\cdot\frac{\pi}{2}}$

1

(iii) $f(a)$ is not defined

$f(x) = \dfrac{1}{x-1}$



To understand explicitly the reasons of discontinuity. Consider the following graph of a function. Note that

f is continuous at $x = 0$ and $x = 4$

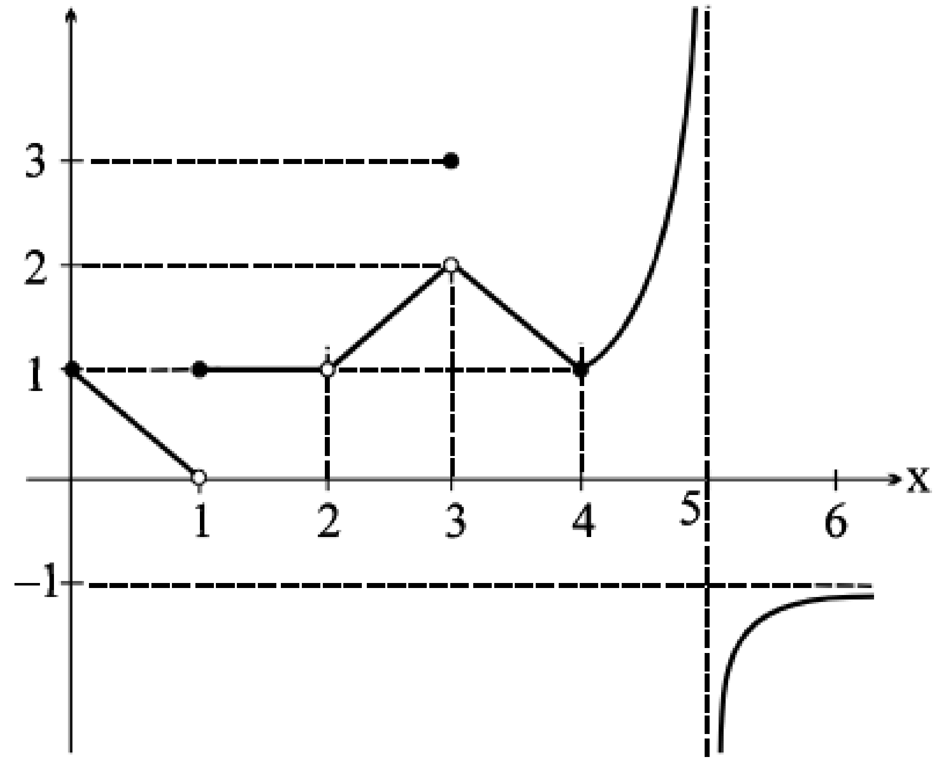
f is discontinuous at $x = 1$ as limit does not exist

f is discontinuous at $x = 2$ as $f(2)$ is not defined although limit exist.

f is discontinuous at $x = 3$ as

$$\lim_{x \to 3} f(x) \neq f(3)$$

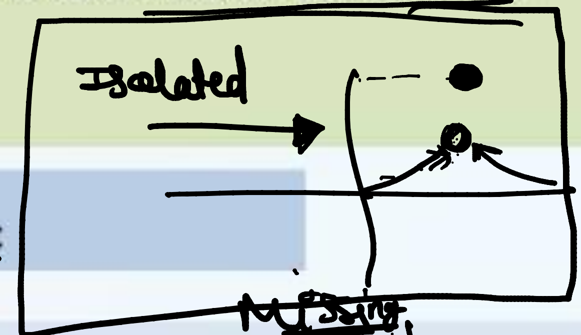
f is discontinuous at $x = 5$ as neither the limit exist nor f is defined at $x = 5$



Type-1 : Removable type of discontinuities

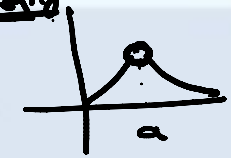
(Type of Discontinuity)

In case $\lim_{x\to a} f(x)$ exists but is not equal to f(a) then the function is said to have a removable discontinuity or discontinuity of the first kind. In this case we can redefine the function such that $\lim_{x\to a} f(x) = f(a)$ & make it continuous at x = a.



Removable type of discontinuity can be further classified as:

(a) **Missing point discontinuity** : Where $\lim_{x\to a} f(x)$ exists but f(a) is not defined



$$\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

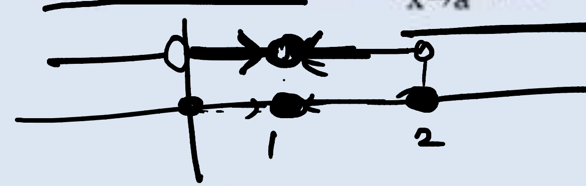
$$\boxed{f(x) = \frac{x^2-4}{x-2} \text{ at } x = 2}$$

$\boxed{R.H.L = L.H.L = 4}$

(b) **Isolated point discontinuity** : Where $\lim_{x\to a} f(x)$ exists & f(a) also exist but; $\lim_{x\to a} f(x) \neq f(a)$

$$f(x) = [x] + [-x]$$

$$= \begin{cases} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{cases}$$



$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

$\sin(\infty)$

Type -2: Non-removable Type of Discontinuities

In case $\lim_{x \to a} f(x)$ does not exist then it is not possible to make the function continuous by redefining it. Such a discontinuity is known as non-removable discontinuity or discontinuity of the 2nd kind.

Non-removable type of discontinuity can be further classified as

(i) **Finite type discontinuity** : In such type of discontinuity left hand limit and right hand limit at a point exists but are not equal.

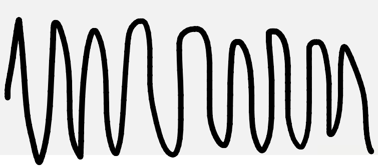
Ex: $\lim_{x \to 2} \frac{[x]}{x}$ $\begin{cases} f(2^+) = 1 \\ f(2^-) = \frac{1}{2} \end{cases}$

(ii) **Infinite type discontinuity** : In such type of discontinuity at least one of the limit viz. LHL and RHL is tending to infinity.

Ex: $f(x) = \frac{1}{x^2}$ at $x = 0$ $\begin{cases} f(0^+) = \infty \\ f(0^-) = \infty \end{cases}$

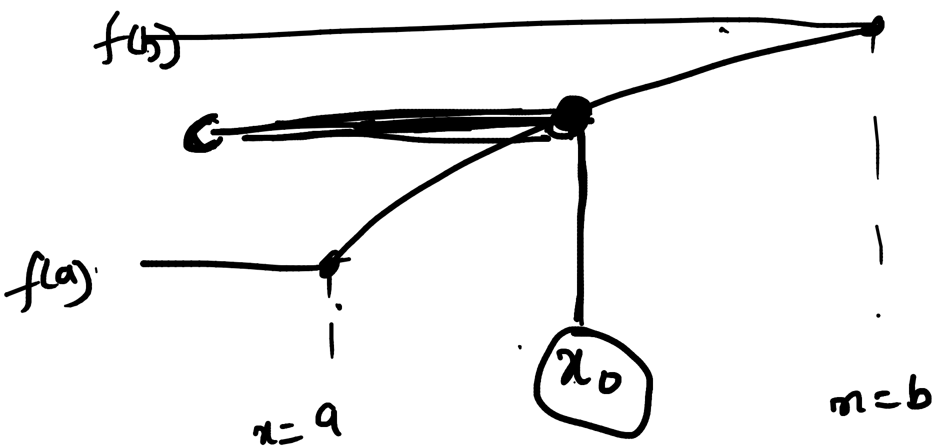
(iii) **Oscillatory** (limits oscillate between two finite quantities)

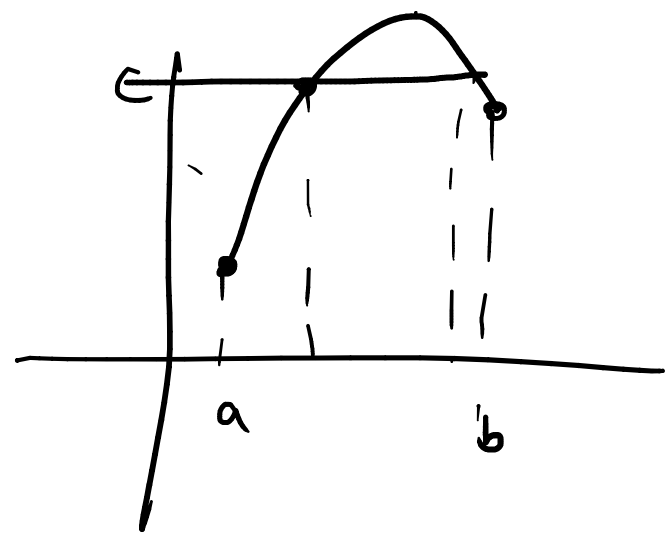
Ex: $\left.\begin{array}{l} f(x) = \sin\frac{1}{x} \\ f(x) = \cos\frac{1}{x} \end{array}\right]$ at $x = 0$



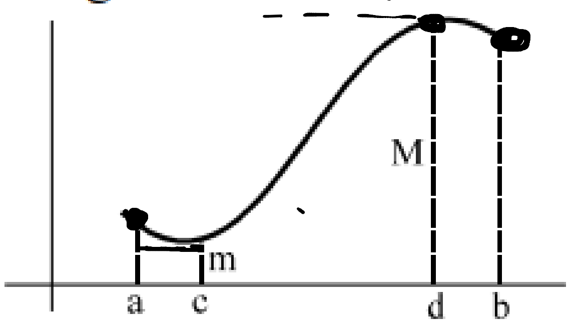
Intermediate value theorem :

If f is continuous on [a, b] and $f(a) \neq f(b)$ then for any value $c \in (f(a), f(b))$, there is at least one number x_0 in (a, b) for which $f(x_0) = c$

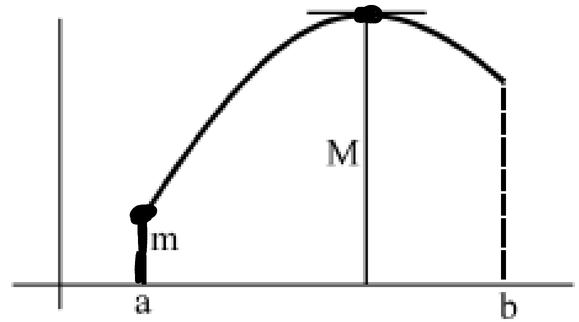




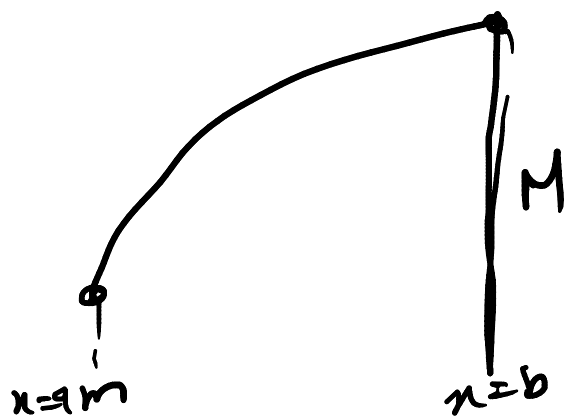
Extreme Value Theorem : If f is continuous on [a, b] then f takes on, a least value of m and a greatest value M on this interval.

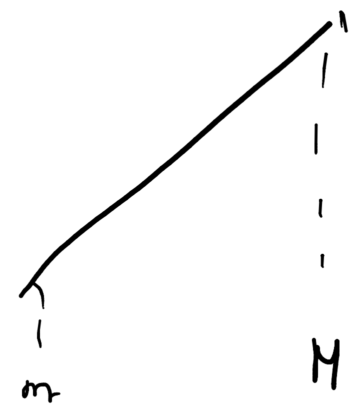


Minimum value 'm' occurs at x = c and maximum value M occurs at x = d. c, d ∈ (a, b)



Minimum value 'm' occurs at the end point x = a and the maximum value M occurs inside the interval



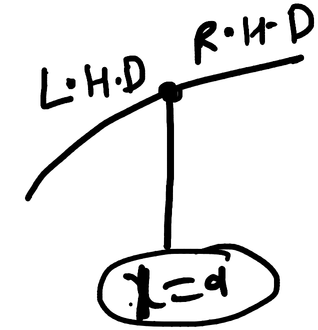


Differentiability

$f(a^+)$

$\lim_{h \to 0} f(a+h) = R \cdot H \cdot L$

$\lim_{h \to 0} f(a-h) \rightarrow$



EXISTENCE OF DERIVATIVE :

Right hand & Left hand Derivatives ;

By definition : $\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$ if it exist

$f'(a^+)$

R·H·D

L·H·D $f'(a^-)$

The right hand derivative of f at x = a denoted by $f'(a^+)$ is defined by :

$$\boxed{f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h)-f(a)}{h}}$$

R·H·D

provided the limit exists & is finite .

The left hand derivative : of f at x = a denoted by $f'(a^-)$ is defined by :

$$\boxed{f'(a^-) = \lim_{h \to 0^+} \frac{f(a-h)-f(a)}{-h}}$$

L·H·D

Provided the limit exists & is finite .

$\boxed{\text{f is said to be derivable at x = a if } f'(a^+) = f'(a^-) = \text{a finite quantity.}}$

$\boxed{R \cdot H \cdot D = L \cdot H \cdot D = \text{a finite quantity}}$

Derivability & Continuity :

Theorem : If a function f is derivable at x = a then f is continuous at x = a.

Proof : Let f is derivable at x = a. Hence

For : $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists .

Also $f(a+h) - f(a) = \left(\frac{f(a+h) - f(a)}{h} \cdot h\right)$ $(h \neq 0)$

Therefore : $\lim_{h \to 0} [(f(a+h) - f(a)] = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} . h = f'(a).0 = 0$

Therefore $\lim_{h \to 0} [(f(a+h) - f(a)] = 0 \Rightarrow \lim_{h \to 0} f(a+h) = f(a) \Rightarrow f$ is continuous at x.

R.H.L = f(a)

For a function f :

(1) Differentiability $\Rightarrow$ Continuity ;

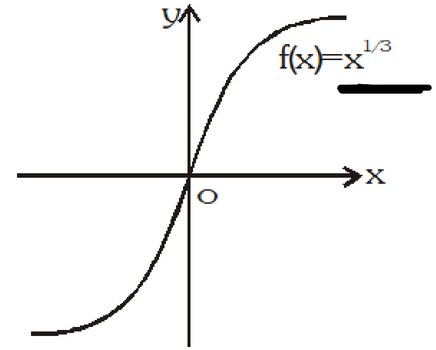
(2) Continuity $\not\Rightarrow$ derivability ;

(3) Non derivibality $\not\Rightarrow$ discontinuous

(4) But discontinuity $\Rightarrow$ Non derivability

Vertical tangent :

(i) If for $y = f(x)$; $f'(a^+) \to \infty$ and $f'(a^-) \to \infty$ or $f'(a^+) \to -\infty$ and $f'(a^-) \to -\infty$ then at $x = a$, $y = f(x)$ has vertical tangent at $x = a$



e.g. (1) $f(x) = x^{1/3}$ has vertical tangent at $x = 0$

$f'(x) = \frac{1}{3} x^{-2/3}$

$= \frac{1}{3x^{2/3}}$

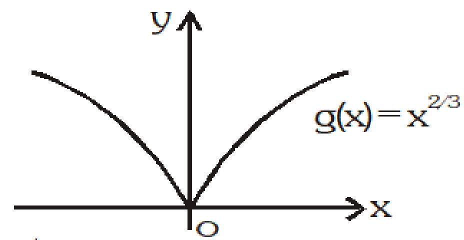
since $f'(0^+) \to \infty$ and $f'(0^-) \to \infty$ hence f(x) is not differentiable at $x = 0$

$g(x) = x^{2/3}$

(2) $g(x) = x^{2/3}$ does'nt have vertical tangent at $x = 0$

$= \frac{2}{3x^{1/3}}$

since $g'(0^+) \to \infty$ and $g'(0^-) \to -\infty$ hence g(x) is not differentiable at $x = 0$.

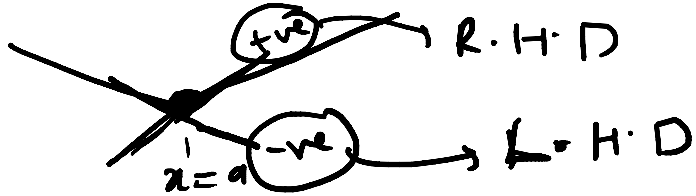


(ii) If a function has vertical tangent at $x = a$ then it is non differentiable at $x = a$.

$y = f(x)$

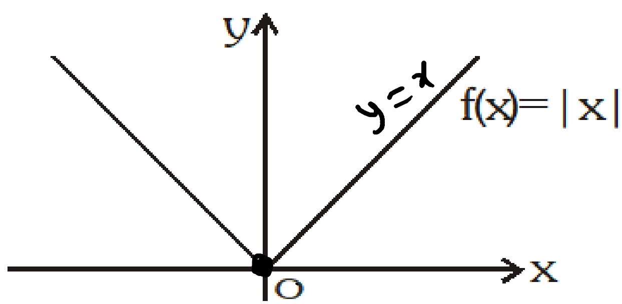
L.H.D, R.H.D at $x = a$ → ∞ → $-\infty$ (Vertical Tangent)

Corner

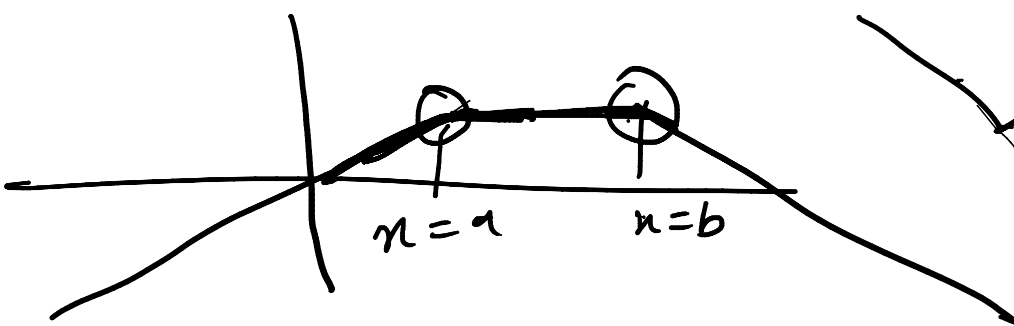


→ If a function f(x) does not have a unique tangent ($p \neq q$) but is continuous at $x = a$, it geometrically implies a sharp corner at $x = a$. Note that p and q may not be finite, where $p = f'(a^+)$ and $q = f'(a^-)$

R·H·D L·H·D



(does not have unique tangent) $p = 1$ (R·H·D), $q = -1$ (L·H·D)



THEOREM : If f (x) and g (x) both are derivable at x = a, then

1. $f(x) \pm g(x)$ will be differentiable at $x = a$.
2. $f(x) \cdot g(x)$ will be differentiable at $x = a$.
3. $\dfrac{f(x)}{g(x)}$ will be differentiable at $x = a$ if $g(a) \neq 0$.