Ellipse

Problem Solving (JEE Mains)

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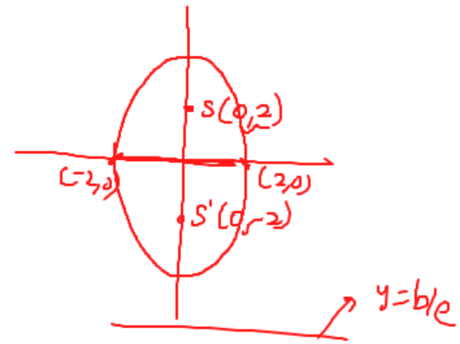
JEE Mains Problems

An ellipse, with foci at (0, 2) and (0, – 2) and minor axis of length 4, passes through which of the following points? **(2019 Main, 12 April II)**

(a) $(\sqrt{2}, 2)$ (b) $(2, \sqrt{2})$

(c) $(2, 2\sqrt{2})$ (d) $(1, 2\sqrt{2})$



$be = 2$ $a = 2$

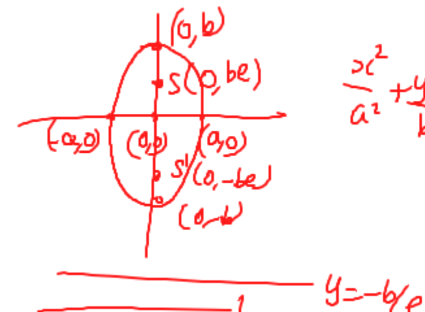
$a^2 = b^2 - b^2e^2$

$4 = b^2 - 4 \Rightarrow b^2 = 8$

$a^2 = 4$

eqn: $\frac{x^2}{4} + \frac{y^2}{8} = 1$

** $\boxed{2x^2 + y^2 = 8}$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$b > a$

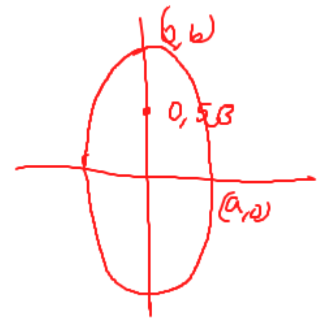
$\boxed{a^2 = b^2(1-e^2)}$

$\cancel{b^2 = a^2(1-e^2)}$ $\quad L.R = \frac{2b^2}{a}$ (H.E)

JEE Mains Problems

In an ellipse, with centre at the origin, if the difference of the lengths of major axis and minor axis is 10 and one of the foci is at $(0, 5\sqrt{3})$, then the length of its latus rectum is **(2019 Main, 8 April I)**

(a) 5 (b) 10 (c) 8 (d) 6



length of major axis = 2b

" " minor axis = 2a

Vertical Ellipse

$$L.R = \frac{2a^2}{b} = \frac{2 \times 25}{10} = \frac{50}{10} = 5$$

$2b - 2a = 10$

$\boxed{b - a = 5}$ — (1)

$be = 5\sqrt{3}$

$a^2 = b^2(1-e^2)$

$a^2 = b^2 - b^2e^2$

$b^2 - a^2 = b^2e^2$

$(b-a)(b+a) = b^2e^2 \Rightarrow 5(b+a) = 75$

$b + a = 15$

$b - a = 5$

$b + a = 15$

$2b = 20$

$b = 10$

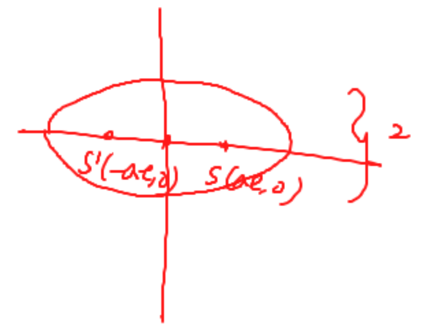
$a = 5$

JEE Mains Problems

Let the length of the latus rectum of an ellipse with its major axis along X-axis and centre at the origin, be 8. If the distance between the foci of this ellipse is equal to the length of its minor axis, then which one of the following points lies on it? **(2019 Main, 11 Jan II)**

(a) $(4\sqrt{2}, 2\sqrt{3})$ (b) $(4\sqrt{3}, 2\sqrt{2})$

(c) $(4\sqrt{2}, 2\sqrt{2})$ (d) $(4\sqrt{3}, 2\sqrt{3})$



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$\frac{2b^2}{a}=8$ — (1) $\Rightarrow \boxed{b^2=4a}$

$\frac{x^2}{64}+\frac{y^2}{32}=1$

$SS'=\boxed{2ae=2b}$ — (2)

$\boxed{x^2+2y^2=64}$

$\boxed{ae=b}$

$b^2=a^2(1-e^2)$

~~48+16=64~~

$b^2=a^2-a^2e^2$

$4a=a^2-b^2 \Rightarrow 4a=a^2-4a \Rightarrow a^2=8a \Rightarrow a=8 \Rightarrow b^2=4\times 8 \Rightarrow b^2=32$

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JEE Mains Problems

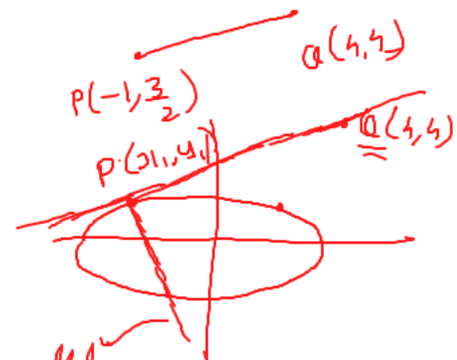
If the normal to the ellipse $3x^2 + 4y^2 = 12$ at a point P on it is parallel to the line, $2x + y = 4$ and the tangent to the ellipse at P passes through $Q(4, 4)$ then PQ is equal to

(2019 Main, 12 April I)

(a) $\frac{5\sqrt{5}}{2}$ ✓ (b) $\frac{\sqrt{61}}{2}$

(c) $\frac{\sqrt{221}}{2}$ (d) $\frac{\sqrt{157}}{2}$

$PQ = \sqrt{5^2 + \left(4 - \frac{3}{2}\right)^2} = \sqrt{25 + \frac{25}{4}} = \sqrt{\frac{125}{4}}$



$\Rightarrow \frac{x^2}{4} + \frac{y^2}{3} = 1$

$2x + y = 4 \Rightarrow y = -2x + 4$

$m_N = -2$

$m_T = \frac{1}{2}$

Eqn of tangent $\equiv$

$y - 4 = \frac{1}{2}(x - 4)$

$2y - 8 = x - 4$

eqn of tangent at $P \equiv$ $\boxed{1x - 2y + 4 = 0}$ — (1)

eqn of tangent at $P \equiv$ $T = 0$

$3xx_1 + 4yy_1 = 12$

$\boxed{3xx_1 + 4yy_1 - 12 = 0}$ — (2)

Same lines.

$\frac{3x_1}{1} = \frac{4y_1}{-2} = \frac{-12}{4} = -3$

$x_1 = -1, \; y_1 = \frac{3}{2}$

JEE Mains Problems

The tangent and normal to the ellipse $3x^2 + 5y^2 = 32$ at the point $P(2, 2)$ meets the X-axis at Q and R respectively. Then, the area (in sq units) of the ΔPQR is

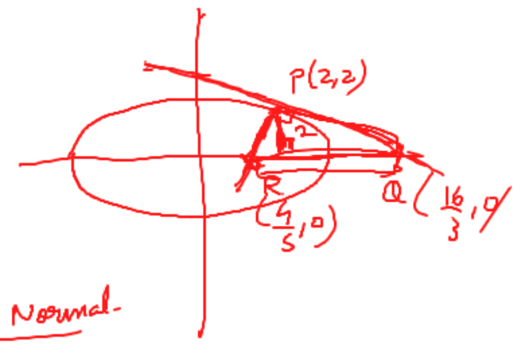
(2019 Main, 10 April II)

(a) $\frac{16}{3}$ (b) $\frac{14}{3}$ (c) $\frac{34}{15}$ (d) $\frac{68}{15}$

$RQ = \frac{16}{3} - \frac{4}{5} = \frac{80-12}{15} = \frac{68}{15}$

base $= \frac{68}{15}$, height $= 2$

Area $= \frac{1}{2} \times 2 \times \frac{68}{15}$



$\frac{x^2}{32/3} + \frac{y^2}{32/5} = 1$

Eqn of tangent = $T = 0$

$3 \times x x_1 + 5 \times y y_1 = 32$

$6x + 10y = 32.$

$\boxed{3x + 5y = 16}$ put $y = 0 \Rightarrow 3x = 16$

$x = 16/3$

$m_T = -\frac{3}{5}$

Eqn of Normal.

$m_N = 5/3$. $P(2,2)$

$y - 2 = \frac{5}{3}(x-2)$ put $y=0$

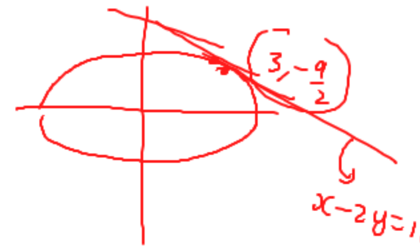
$-2 = \frac{5}{3}(x-2) \Rightarrow x = -\frac{6}{5} + 2 = \frac{4}{5}$

JEE Mains Problems

If the line $x - 2y = 12$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $\left(3, \frac{-9}{2}\right)$, then the length of the latusrectum of the ellipse is **(2019 Main, 10 April I)**

$LR = 2b^2/a$

(a) $8\sqrt{3}$ (b) 9 (c) 5 (d) $12\sqrt{2}$



$$\frac{x \times 3}{a^2} + \frac{y \times (-9/2)}{b^2} = 1$$

$$\frac{3x}{a^2} - \frac{9y}{2b^2} - 1 = 0 \quad \text{--- (1)}$$

$$x - 2y - 12 = 0 \quad \text{--- (2)}$$

same lines

$$\frac{3/a^2}{1} = \frac{-9/2b^2}{-2} = \frac{-1}{-12}$$

$$\frac{3}{a^2} = \frac{1}{12} \Rightarrow a^2 = 36, \quad a = 6$$

$$\frac{9}{4b^2} = \frac{1}{12}, \quad b^2 = 27$$

$$LR = \frac{2b^2}{a} = \frac{2 \times 27}{6} = 9$$

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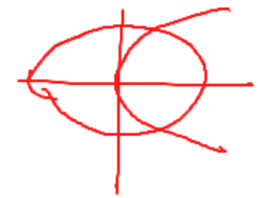
JEE Mains Problems

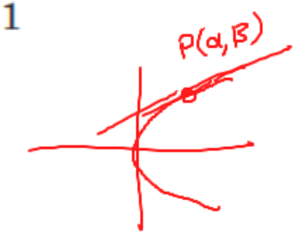
$\boxed{\beta^2=\alpha}$

If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to (2019 Main, 9 April II)

(a) $\sqrt{2}+1$ (b) $\sqrt{2}-1$ (c) $2\sqrt{2}+1$ (d) $2\sqrt{2}-1$



$$\frac{x^2}{1}+\frac{y^2}{1/2}=1$$



$T=0$ $\quad y^2=x$

$yy_1=\frac{1}{2}(x+x_1)$

$2yy_1=x+x_1$

$2\beta y=x+\alpha$

$$\boxed{y=\frac{1}{2\beta}x+\frac{\alpha}{2\beta}}$$

is a tangent of ellipse

Condition of tangency of ellipse.

$c^2=a^2m^2+b^2$

$$\left(\frac{\alpha}{2\beta}\right)^2=1\times\left(\frac{1}{2\beta}\right)^2+\frac{1}{2}$$

$$\frac{\alpha^2}{4\beta^2}=\frac{1}{4\beta^2}+\frac{1}{2} \Rightarrow \frac{\alpha^2}{4\alpha}=\frac{1}{4\alpha}+\frac{1}{2}$$

$$\alpha=\frac{1}{\alpha}+2 \Rightarrow \alpha^2-2\alpha-1=0$$

$$\alpha=\frac{2\pm\sqrt{4+4}}{2}=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$

JEE Mains Problems

If the tangents on the ellipse $4x^2 + y^2 = 8$ at the points $(1, 2)$ and (a, b) are perpendicular to each other, then a^2 is equal to (2019 Main, 8 April I)

(a) $\frac{128}{17}$ (b) $\frac{64}{17}$ (c) $\frac{4}{17}$ (d) $\frac{2}{17}$

$4x^2+y^2=8$

$\frac{x^2}{2}+\frac{y^2}{8}=1$

eqn of tangent is $T=0$

$(1,2)$ $4x \times x \times 1 + y \times 2 = 8$

$4x+2y=8$

$2x+y=4$

$\boxed{m_1=-2}$

(a,b) $4xa+yb=8$

$m_2=\frac{-4a}{b}$

$m_1 \times m_2 = -1$

$-2 \times \frac{-4a}{b} = -1 \Rightarrow \frac{a}{b} = \frac{1}{-8} \Rightarrow \boxed{b=-8a}$

$4a^2+b^2=8$

$4a^2+(-8a)^2=8$

$4a^2+64a^2=8$

$68a^2=8$

$a^2=\frac{8}{68}=\frac{2}{17}$

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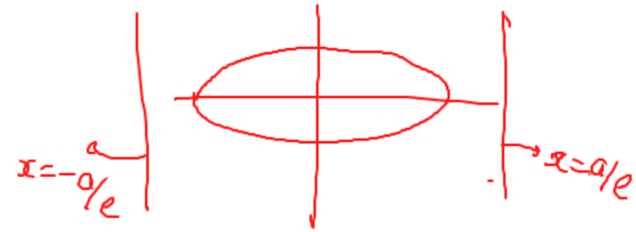
JEE Mains Problems

The eccentricity of an ellipse whose centre is at the origin is 1/2. If one of its directrices is $x = -4$, then the equation of the normal to it at $P\left(1, \frac{3}{2}\right)$ is $m_N = 2$ **(2017 Main)**

(a) $2y - x = 2$
(b) $4x - 2y = 1$
(c) $4x + 2y = 7$
(d) $x + 2y = 4$



$P(1, \frac{3}{2})$ $m_N = 2$

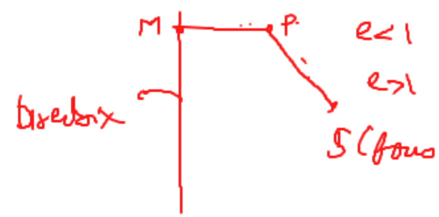
$y - \frac{3}{2} = 2(x-1)$

$2y - 3 = 4x - 4$

$\boxed{4x - 2y = 1}$

$\frac{SP}{PM} = e$

$e = 1$ (parabola)
$e < 1$ (ellipse)
$e > 1$ (hyperbola)



$e = 1/2$

$\frac{a}{e} = 4$

$a = 4e = 4 \times \frac{1}{2} = 2$

$a = 2$

$e = 1/2$

$b^2 = a^2(1 - e^2)$

$b^2 = 4(1 - \frac{1}{4})$

$b^2 = 3$

$a^2 = 4$

$\frac{x^2}{4} + \frac{y^2}{3} = 1$

$3x^2 + 4y^2 = 12$

On differentiation

$6x + 8y \times \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-6x}{8y} = \frac{-3x}{4y} = \frac{-3}{4 \times \frac{3}{2}} = \frac{-1}{2}$

$m_T = -1/2 \Rightarrow m_N = 2$

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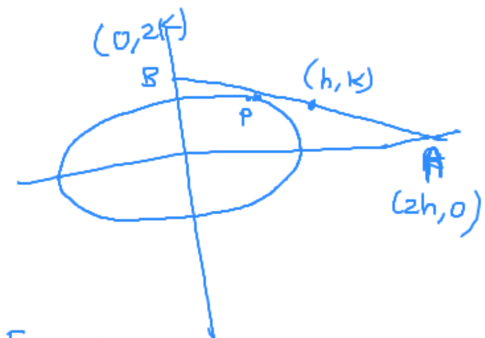
JEE Mains Problems

If tangents are drawn to the ellipse $x^2+2y^2=2$ at all points on the ellipse other than its four vertices, then the mid-points of the tangents intercepted between the coordinate axes lie on the curve **(2019 Main, 11 Jan I)**



(a) $\frac{x^2}{4}+\frac{y^2}{2}=1$ (b) $\frac{1}{4x^2}+\frac{1}{2y^2}=1$

$\frac{x^2}{2}+\frac{y^2}{1}=1$

(c) $\frac{x^2}{2}+\frac{y^2}{4}=1$ (d) $\frac{1}{2x^2}+\frac{1}{4y^2}=1$

Eqn of AB ≡

$\frac{x}{2h}+\frac{y}{2k}=1$

$\boxed{kx+hy=2hk}$ eqn of AB

AB is the tangent of ellipse.

$hy=-kx+2hk$

$y=\left(-\frac{k}{h}\right)x+2k$

$\frac{x^2}{2}+\frac{y^2}{1}=1$

$a^2=2,\ b^2=1$

$m=-\frac{k}{h},\ c=2k$

$c^2=a^2m^2+b^2$

$4k^2=2\frac{k^2}{h^2}+1 \Rightarrow 4=\frac{2}{h^2}+\frac{1}{k^2} \Rightarrow \boxed{\frac{1}{2h^2}+\frac{1}{4k^2}=1}$ $y=mx+c$

$\boxed{\frac{1}{2x^2}+\frac{1}{4y^2}=1}$

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JEE Mains Problems

If the area of the triangle whose one vertex is at the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and Y-axis, is 250 sq units, then a value of 'a' is

(2019 Main, 11 Jan, II)

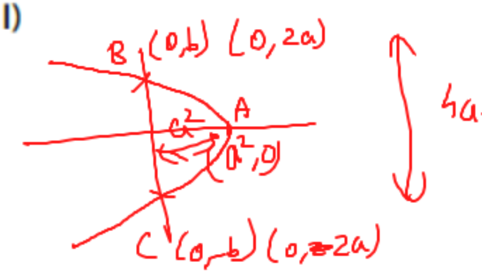
(a) $5\sqrt{5}$ (b) 5

(c) $5(2^{1/3})$ (d) $(10)^{2/3}$

$y^2 = -4(x-a^2) = 0$

$y^2 = -4x = 0$

$V(a^2, 0)$



$\text{Area} = \frac{1}{2} \times 2b \times a^2 = ba^2$

$\text{Area} = \frac{1}{2} \times 4a \times a^2 = 2a^3$

$y^2 = -4(x-a^2)$

$y^2 = 4a^2$

$y = \pm 2a$

y-axis

$x = 0$

$2a^3 = 250$

$a^3 = 125 \Rightarrow a = 5\sqrt{5}$

JEE Mains Problems

If the area of the triangle whose one vertex is at the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and Y-axis, is 250 sq units, then a value of 'a' is

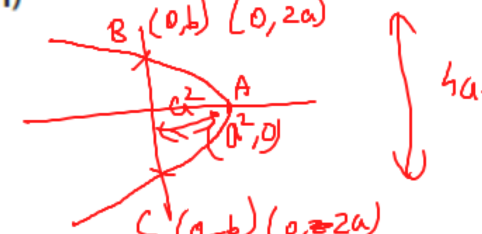
(2019 Main, 11 Jan, II)

(a) $5\sqrt{5}$ ~~(b)~~ 5
(c) $5(2^{1/3})$ (d) $(10)^{2/3}$

$y^2 = -4(x - a^2) = 0$

$y^2 = -4x = 0$

$V(a^2, 0)$



$\text{Area} = \frac{1}{2} \times 2b \times a^2 = ba^2$

$\text{Area} = \frac{1}{2} \times 4a \times a^2 = 2a^3$

$y^2 = -4(x - a^2)$

y-axis

$x = 0$

$y^2 = 4a^2$

$y = \pm 2a$

$2a^3 = 250$

$a^3 = 125 \Rightarrow$ ~~$a = 5\sqrt{5}$~~

$a = 5$

JEE Mains Problems

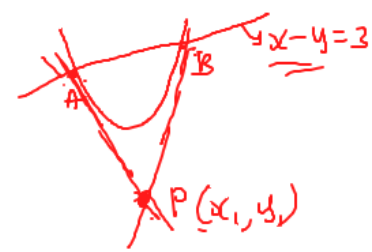
The tangents to the curve $y = (x-2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point

(2019 Main, 12 April II)

(a) $\left(\frac{5}{2}, 1\right)$ (b) $\left(-\frac{5}{2}, -1\right)$ (c) $\left(\frac{5}{2}, -1\right)$ (d) $\left(-\frac{5}{2}, 1\right)$

$y+1 = (x-2)^2$

$X^2 = Y$



w.r.t P, AB is chord of contact

$y+1 = x^2 + 4 - 4x$

$x^2 - y - 4x + 3 = 0$

eqn of C.O.C

T=0

$xx_1 - \frac{1}{2}(y+y_1) - 2(x+x_1) + 3 = 0$

$x(x_1 - 2) - \frac{1}{2}y - \frac{y_1}{2} - 2x_1 + 3 = 0$

$\boxed{(2x_1-4)x - y - y_1 - 4x_1 + 6 = 0}$ C.O.C

$x - y - 3 = 0$

Same

$\frac{2x_1 - 4}{1} = \frac{+1}{+1} = \frac{-y_1 - 4x_1 + 6}{-3}$

$2x_1 - 4 = 1$

$2x_1 = 5$

$x_1 = 5/2$

$1 = \frac{-y_1 - 4x_1 + 6}{-3}$

$-3 = -y_1 - 4x_1 + 6$

$4x_1 + y_1 = 9 \Rightarrow y_1 =$

JEE Mains Problems

$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Equation of a common tangent to the circle, $x^2 + y^2 - 6x = 0$ and the parabola, $y^2 = 4x$, is

(2019 Main, 9 Jan, I)

$C(3,0)$ $\quad r = \sqrt{3^2+0-0} = 3$

(a) $\sqrt{3}y = 3x + 1$ (b) $2\sqrt{3}y = 12x + 1$

(c) $\sqrt{3}y = x + 3$ (d) $2\sqrt{3}y = -x - 12$

$y^2 = 4x \Rightarrow y^2 = 4ax \Rightarrow a = 1$

$y = mx + \frac{a}{m}$

$\boxed{y = mx + \frac{1}{m}}$

tangent of parabola which is also tangent of circle

$m^2x - my + 1 = 0$

$\frac{1}{3}x \mp \frac{1}{\sqrt{3}}y + 1 = 0$

$x \mp \sqrt{3}y + 3 = 0$

$\sqrt{3}y = x + 3$

$m^2x - my + 1 = 0$

$\left|\frac{3m^2+1}{\sqrt{m^4+m^2}}\right| = 3$

$(3m^2+1)^2 = 9(m^4+m^2)$

$9m^4+1+6m^2 = 9m^4+9m^2 \Rightarrow 3m^2 = 1$, $m^2 = 1/3$, $m = \pm\frac{1}{\sqrt{3}}$

JEE Mains Problems

$2y\frac{dy}{dx} = 4b$

$\frac{dy}{dx} = \frac{2b}{y_1}$

If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a, b, c)?

(2019 Main, 10 Jan, I)

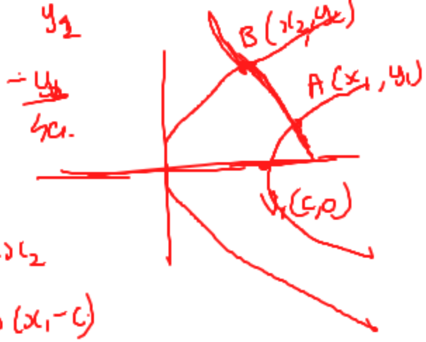
(a) $\left(\frac{1}{2}, 2, 0\right)$ (b) $(1, 1, 0)$

(c) $(1, 1, 3)$ (d) $\left(\frac{1}{2}, 2, 3\right)$

$2y\frac{dy}{dx} = 8a$

$\frac{dy}{dx} = \frac{4a}{y_2}$

$-\frac{dx}{dy} = -\frac{y_2}{4a}$



$y_2^2 = 8ax_2$

$y_1^2 = 4b(x_1 - c)$

$x_2 = \frac{y_2^2}{8a}$, $x_1 = \frac{y_1^2}{4b} + c$

$x_2 - x_1 = \frac{y_2^2}{8a} - \frac{y_1^2}{4b} - c$ — (1)

$m_N(x_1, y_1) = m_N(x_2, y_2) = m_{AB}$

$m_N(x_1, y_1) = m_N(x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$

$-\frac{y_1}{2b} = -\frac{y_2}{4a} = \frac{y_2 - y_1}{x_2 - x_1}$ — (2)