Ellipse & Hyperbola

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$\frac{SP}{PM} = e \quad (e<1)$

Ellipse

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Standard equation of an ellipse referred to its principal axes along the co-ordinate axes is $\boxed{\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1}$.

Where $a > b$ & $b^2 = a^2(1 - e^2) \Rightarrow a^2 - b^2 = a^2 e^2$.

Where e = eccentricity $(0 < e < 1)$.

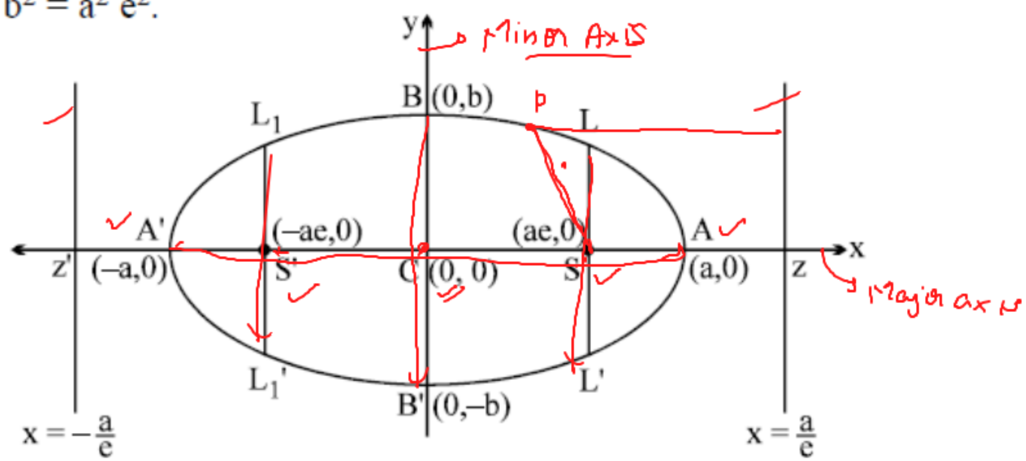
FOCI : $S \equiv (a\,e,\ 0)$ & $S' \equiv (-a\,e,\ 0)$.

EQUATIONS OF DIRECTRICES :

$x = \frac{a}{e}$ & $x = -\frac{a}{e}$.

VERTICES :

$A' \equiv (-a, 0)$ & $A \equiv (a, 0)$.



MAJOR AXIS :

The line segment $A'A$ in which the foci
S' & S lie is of length $2a$ & is called the **major axis** $(a > b)$ of the ellipse. Point of intersection of major axis with directrix is called **the foot of the directrix (z).**

MINOR AXIS :

The y–axis intersects the ellipse in the points $B' \equiv (0, -b)$ & $B \equiv (0, b)$. The line segment $B'B$ of length $2b$ $(b < a)$ is called the **Minor Axis** of the ellipse.

PRINCIPAL AXIS :

The major & minor axis together are called **Principal Axis** of the ellipse.

CENTRE :

The point which bisects every chord of the conic drawn through it is called the **centre** of the conic.

$C \equiv (0, 0)$ the origin is the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

FOCAL CHORD : A chord which passes through a focus is called a **focal chord**.

DOUBLE ORDINATE :
A chord perpendicular to the major axis is called a **double ordinate.**

LATUS RECTUM :
The focal chord perpendicular to the major axis is called the **latus rectum**. Length of latus rectum

$$(LL') = \boxed{\frac{2b^2}{a}} = \frac{(minor\ \ axis)^2}{major\ \ axis} = 2a(1-e^2) = 2e\,(\text{distance from focus to the corresponding directrix})$$

Concepts

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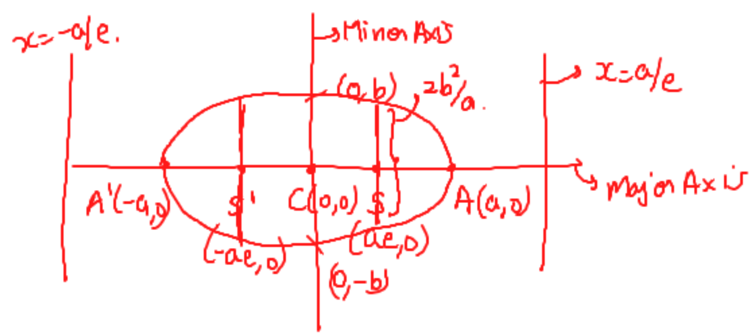


$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \quad a > b$$

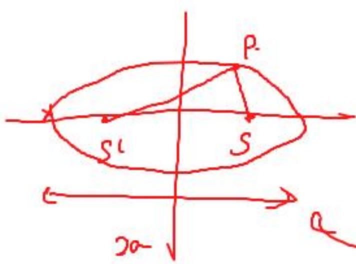
$$\boxed{b^2 = a^2(1-e^2)}$$

Concepts

NOTE :

(i) The sum of the focal distances of any point on the ellipse is equal to the major Axis. Hence distance of focus from the extremity of a minor axis is equal to semi major axis. **i.e. BS = CA.**

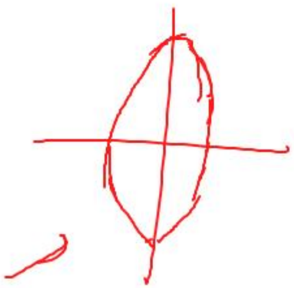
(ii) If the equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & nothing is mentioned then the rule is to assume that $a > b$.



$a > b$

$\boxed{PS + PS' = 2a}$

$\frac{x^2}{4^2}+\frac{y^2}{2^2}=1$ | $\frac{x^2}{2^2}+\frac{y^2}{4^2}=1$



$a < b$

Problems

The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having centre at (0, 3) is : [JEE (Main)]

(1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$ (3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$

$\frac{x^2}{4^2}+\frac{y^2}{3^2}=1$

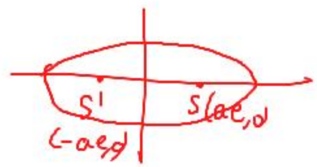
$a=4,\ b=3$

$b^2=a^2(1-e^2)$

$\frac{b^2}{a^2}=1-e^2$

$\boxed{e^2=1-\frac{b^2}{a^2}} \Rightarrow e^2=1-\frac{9}{16}=\frac{7}{16}$

$e=\frac{\sqrt{7}}{4}$



$ae=4\times\frac{\sqrt{7}}{4}=\sqrt{7}$

$S(\sqrt{7},0)$ $S'(-\sqrt{7},0)$

$C(0,3)$

radius $= CS = \sqrt{7+9}=4$

$(x-0)^2+(y-3)^2=4^2$

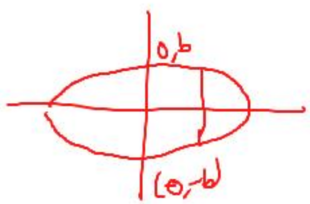
$x^2+y^2+9-6y=16$

$x^2+y^2-6y-7=0$

Problems

If LR of an ellipse is half of its minor axis, then its eccentricity is -

(A) $\frac{3}{2}$ (B) $\frac{2}{3}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{2}}{3}$



$LR = \frac{2b^2}{a}$

Length of minor axis = $2b$

half of minor axis = b

$\frac{2b^2}{a} = b \Rightarrow \boxed{\frac{b}{a} = \frac{1}{2}}$

$b^2 = a^2(1-e^2)$

$1-e^2 = \frac{b^2}{a^2}$

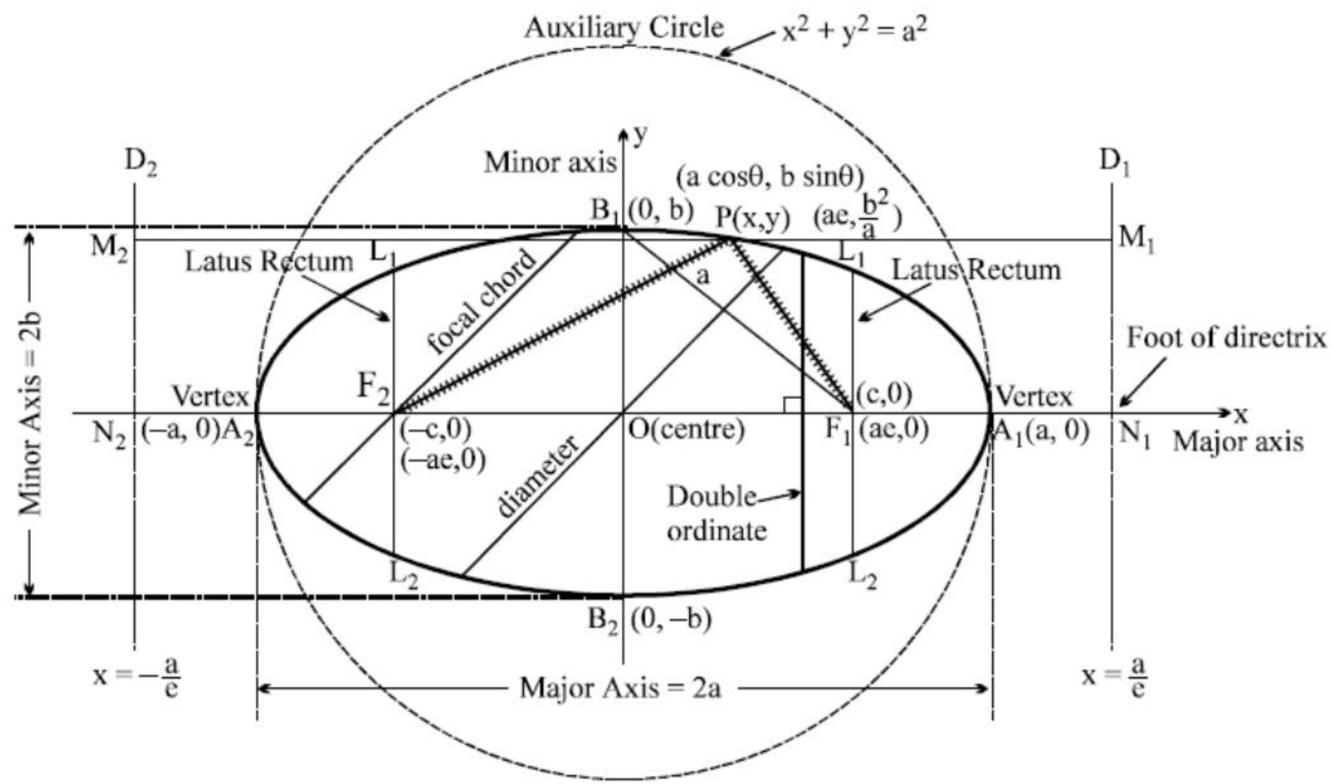
$e^2 = 1 - \frac{b^2}{a^2}$

$= 1 - \frac{1}{4}$

$= \frac{3}{4}$

$e = \sqrt{3}/2$

ELLIPSE AT A GLANCE



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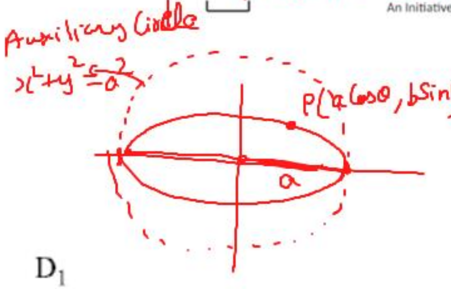
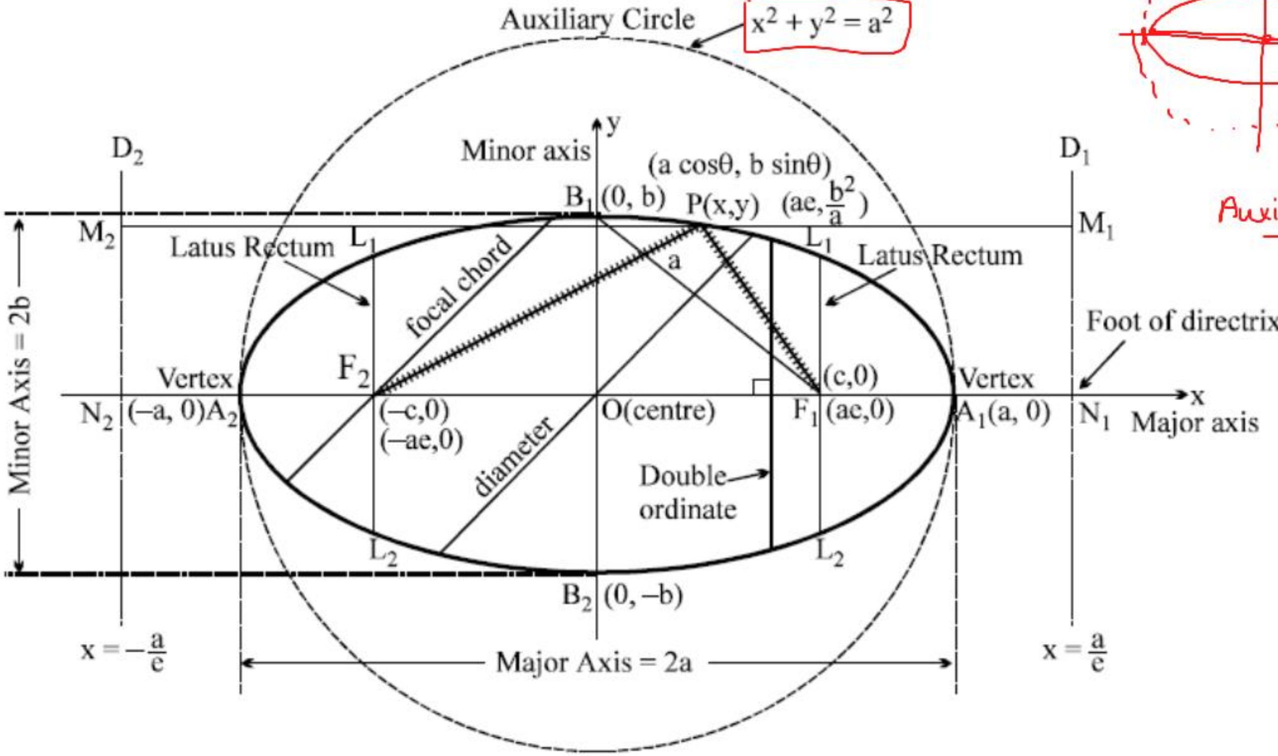


Auxiliary Circle



POSITION OF A POINT w.r.t. AN ELLIPSE :

The point $P(x_1, y_1)$ lies outside, inside or on the ellipse according as ; $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1 > < \text{ or } = 0$.

$S_1 > 0$ outside
$S_1 = 0$ on the curve
$S_1 < 0$ inside.

Circle/Parabola/Ellipse

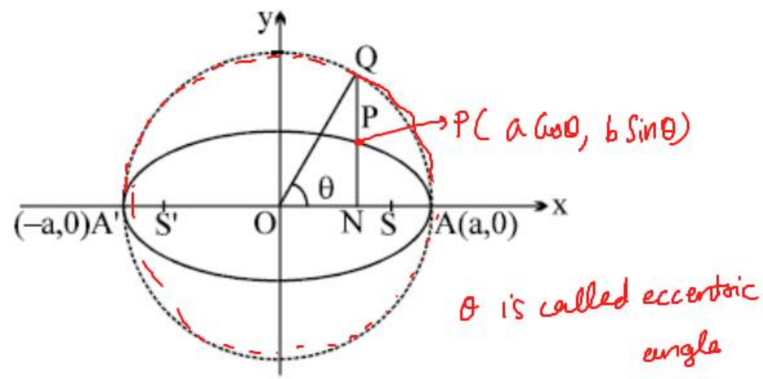
$\boxed{S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1}$ (x_1, y_1)

$\boxed{S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1}$

AUXILIARY CIRCLE / ECCENTRIC ANGLE :

A circle described on major axis as diameter is called the **auxiliary circle**.

Let Q be a point on the auxiliary circle $x^2 + y^2 = a^2$ such that QP produced is perpendicular to the x-axis then P & Q are called as the **CORRESPONDING POINTS** on the ellipse & the auxiliary circle respectively 'θ' is called the **ECCENTRIC ANGLE** of the point P on the ellipse $(0 \le \theta < 2\pi)$.



PARAMETRIC REPRESENTATION :

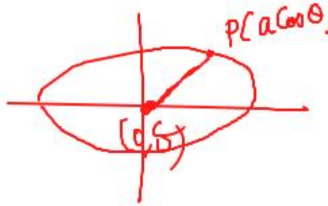
The equations $x = a \cos \theta$ & $y = b \sin \theta$ together represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Where θ is a parameter. Note that if $P(\theta) \equiv (a \cos \theta, b \sin \theta)$ is on the ellipse then ; $Q(\theta) \equiv (a \cos \theta, a \sin \theta)$ is on the auxiliary circle.

Problems

If the distance of a point on the ellipse $\frac{x^2}{6}+\frac{y^2}{2}=1$ from the centre is 2, then the eccentric angle is-

(A) $\pi/3$ (B) $\pi/4$ (C) $\pi/6$ (D) $\pi/2$



$CP = 2$

$\sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = 2$

$6\cos^2\theta + 2\sin^2\theta = 4$

$4\cos^2\theta + 2 = 4$

$4\cos^2\theta = 2$

$\cos^2\theta = \frac{1}{2}$

$\cos\theta = 1/\sqrt{2}$

$\theta = \pi/4$

LINE AND AN ELLIPSE :

The line $y = mx + c$ meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in two points real, coincident or imaginary according as c^2 is $<$ $=$ or $>$ $a^2m^2 + b^2$.

$c^2 < a^2m^2+b^2$ (secant) $c^2 > a^2m^2+b^2$ (niether)

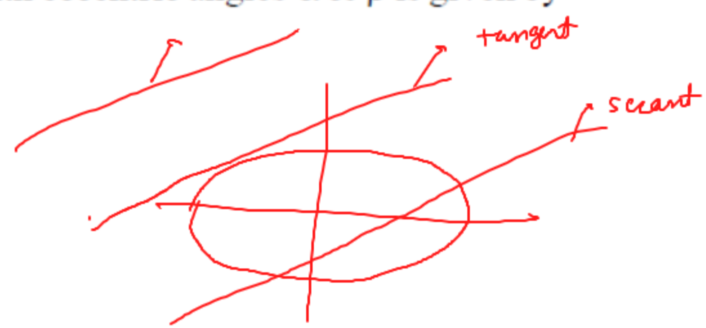
$\boxed{c^2 = a^2m^2+b^2 \ \text{(tangent)}}$

$c = \pm\sqrt{a^2m^2+b^2}$

Hence $y = mx + c$ is tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 + b^2$.

The equation to the chord of the ellipse joining two points with eccentric angles α & β is given by

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}.$$



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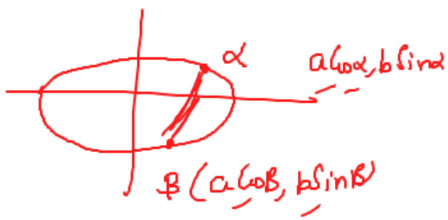
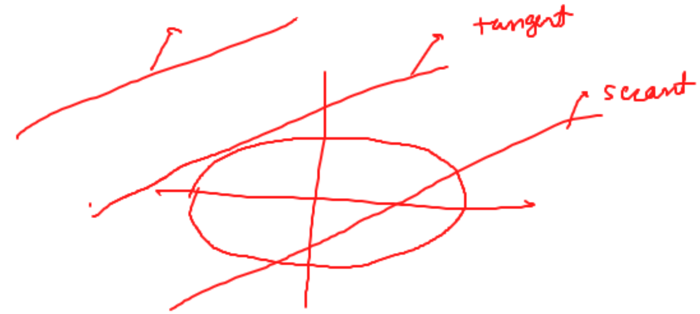
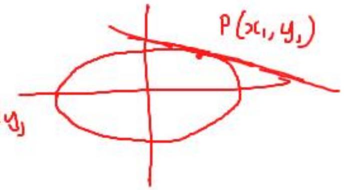




$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$x^2 \to x x_1$
$y^2 \to y y_1$
$2x \to x+x_1$
$2y \to y+y_1$
$2xy \to x_1 y + x y_1$



$T=0$

TANGENTS :

(i) $\boxed{\frac{x x_1}{a^2}+\frac{y y_1}{b^2}=1}$ is tangent to the ellipse at (x_1, y_1).

$c^2 = a^2m^2+b^2 \Rightarrow c = \pm\sqrt{a^2m^2+b^2}$

(ii) $y = mx \pm \sqrt{a^2m^2+b^2}$ is tangent to the ellipse for all values of m.

Note that there are two tangents to the ellipse having the same m, i.e. there are two tangents parallel to any given direction.

$(x_1, y_1) \to (a\cos\theta, b\sin\theta)$

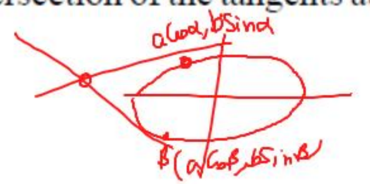
$\frac{x\, a\cos\theta}{a^2}+\frac{y\, b\sin\theta}{b^2}=1$

(iii) $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ is tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$.

(iv) The eccentric angles of point of contact of two parallel tangents differ by π. Conversely if the difference between the eccentric angles of two points is p then the tangents at these points are parallel.

(v) Point of intersection of the tangents at the point α & β is $\boxed{a\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ b\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}}$.

$\boxed{a t_1 t_2,\ a(t_1+t_2)}$



Problems

For what value of λ does the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$.

$y = mx + c$

$m = 1, \quad c = \lambda$

$\frac{x^2}{16} + \frac{y^2}{9} = 1$

$a = 4, \quad b = 3$

$c^2 = a^2 m^2 + b^2$

$\lambda^2 = 16 \times 1 + 9$

$\lambda^2 = 25$

$\lambda = \pm 5$

NORMALS :

(i) Equation of the normal at (x_1, y_1) is $\boxed{\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2.}$

(ii) Equation of the normal at the point $(a\cos\theta, b\sin\theta)$ is ; $ax\sec\theta - by\operatorname{cosec}\theta = (a^2 - b^2)$.

(iii) Equation of a normal in terms of its slope 'm' is $\boxed{y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}}$.

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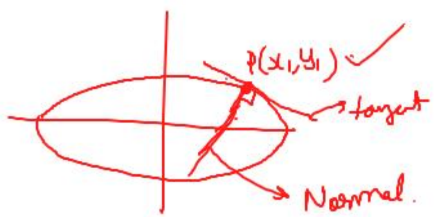

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x/a^2}{y/b^2} = \frac{-b^2x_1}{a^2y_1} \qquad m_N = \frac{a^2y_1}{b^2x_1}$$

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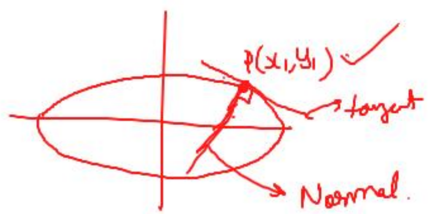

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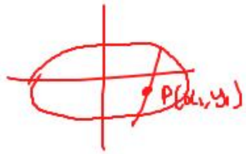
Eqn of chord with given middle point

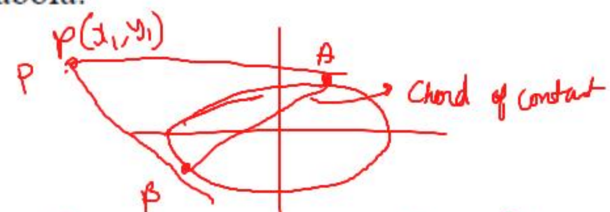


$$T = S_1$$

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}\cancel{-1} = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}\cancel{-1}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Chord of contact, pair of tangents, chord with a given middle point, are to be interpreted as they are in parabola.



Eqn of chord of contact AB $\equiv$ $T = 0$

$$\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$$

Combined eqn of pair of tangents is $SS_1 = T^2$

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$