Limits

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CONCEPT OF LIMITS :

Suppose f(x) is a real-valued function and c is a real number. The expression $\lim_{x \to c} f(x) = L$ means that f(x) can be as close to L as desired by making x sufficiently close to c. In such a case, we say that limit of f, as x approaches c, is L. Consider $f(x) = x - 1$ as x approaches 2.

L·H·S → $x = 2$ ← R·H·S

f(1.9)	f(1.99)	f(1.999)	f(2)	f(2.001)	f(2.01)	f(2.1)
0.9	0.99	0.999	→1←	1.001	1.01	1.1

$\lim_{x \to a} f(x) = f(a)$

As x aproaches 2, f(x) approaches 1 and hence we have $\boxed{\lim_{x \to 2} f(x) = 1}$.

REMEMBER
$\lim_{x \to a} \Rightarrow x \neq a$

Left-and Right-Hand Limits :

R.H.L Right hand limit of a function is that value of $f(x)$ which function tends as x moves from right to number 'a' that is RHL $= \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$ (where $h > 0$)

L.H.L Left hand limit of a function is that value of $f(x)$ which function tends x moves from left to number 'a'. that is LHL $= \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$ (where $h > 0$)

Existence of Limit :

L·H·L

If follows from the discussions made in the previous two sections that $\lim_{x \to a} f(x)$ exists if $\lim_{x \to a^-} f(x)$ and R·H·L $\lim_{x \to a^+} f(x)$ exists and both are equal.

Thus, $\lim_{x \to a} f(x)$ exists $\Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

$$\boxed{\text{L·H·L} = \text{R·H·L} = \text{finite}}$$

INDETERMINATE FORMS :

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Indeterminant forms are $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^{∞} , 0^0 and ∞^0

Limit

$(-.00001) \times \infty = -\infty$

$(.00001) \times \infty = \infty$

$\infty^{\infty} =$

$x + x = 2x$

Note : ~~(i)~~ Here 0,1 are not exact, infact both are aproaching to their corresponding values.

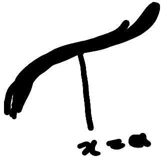
~~(ii)~~ We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

(a) $\infty + \infty \to \infty$ (b) $\infty \times \infty \to \infty$ (c) $\infty^{\infty} \to \infty$ (d) $0^{\infty} \to 0$

EVALUATION OF ALGEBRAIC LIMITS :

Direct Substitution Method :

Consider the following limits : $\lim_{x \to a} f(x)$ and f(x) is continuous at x = a



then we say that $\boxed{\lim_{x \to a} f(x) = f(a)}$

Evaluate : (i) $\lim_{x \to 1} 3x^2 + 4x + 5$

$= 3(1)^2 + 4(1) + 5$

$= \underline{\underline{12}}$

(ii) $\lim_{x \to 2} \dfrac{x^2 - 4}{x + 3} = 0$

(ii) $\dfrac{(2)^2 - 4}{2 + 3} = \underline{\underline{0}}$

When $x \to \infty$

In this case expression should be expressed as a function of 1/x and then after removing indeterminant form, (If it is there) replace 1/x by 0.

3) Find the value of $\lim\limits_{x\to\infty} \dfrac{2x^3+4x^2+5}{9x^3+4x^2+7}$ $\left(\dfrac{\infty}{\infty}\right)$

$\dfrac{1}{x} \to 0$

$$\lim_{x\to\infty} \frac{2x^3+4x^2+5}{9x^3+4x^2+7} = \lim_{x\to\infty} \frac{\cancel{x^3}\left(2+\frac{4}{x}+\frac{5}{x^3}\right)}{\cancel{x^3}\left(9+\frac{4}{x}+\frac{7}{x^3}\right)}$$

$$= \frac{2}{9} \quad \left(\frac{2}{9}\right)$$

9) The value of $\lim_{x \to \infty} \frac{1^2 + 2^2 + 3^2 + \dots + x^2}{x^3}$ is -

(A) 1/2 ~~(B)~~ 1/3 (C) 1 (D) Does not exist

$$1^2 + 2^2 + 3^2 - - + x^2 = \frac{x(x+1)(2x+1)}{6}$$

$$\lim_{x \to \infty} \frac{1^2 + 2^2 + 3^2 + \dots + x^2}{x^3} = \lim_{x \to \infty} \frac{x(x+1)(2x+1)}{6x^3}$$

$$\lim_{x \to \infty} \frac{x(2x+1)(x+1)}{6x^3}$$

$$= \lim_{x \to \infty} \frac{(1+1/x)(2+1/x)}{6} = 1/3$$

$$\lim_{n \to \infty} \frac{\cancel{x^3}\left[1 \cdot \left(2 + \cancel{\frac{1}{x}}^{0}\right)\left(1 + \cancel{\frac{1}{x}}^{0}\right)\right]}{6 \cancel{x^3}}$$

$$\frac{\cancel{2}}{\cancel{6}} = \frac{1}{3}$$

Factorisation :

If f(x) is of the form $\frac{h(x)}{g(x)}$ and of indeterminate form $\frac{0}{0}$ then this form is removed by factorising g(x) and h(x) and cancel the common factors, then put the value of x.

The value of $\lim_{x \to 2} \left(\frac{2x^2 - 7x + 6}{5x^2 - 11x + 2} \right)$ is -

$2 \cdot 4 - 14 + 6$

(A) 1/2 (B) 1/4 (C) 1/9 (D) Does not exist

$$\lim_{x \to 2} \frac{\cancel{(x-2)}(2x-3)}{\cancel{(x-2)}(5x-1)} = \lim_{x \to 2} \frac{(2x-3)}{(5x-1)}$$

$$= \frac{1}{9}$$

Rationalisation :

In this method we rationalise the factor containing the square root or cube root and simplify and then put the value of x.

The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{(1+x^2)} - \sqrt{1-x^2}}{x^2}$ is- $\left(\dfrac{0}{0}\right)$

(A) 1 (B) 2 (C) 3 (D) Does not exist

$$\lim_{x \to 0} \frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2} \cdot \frac{\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)}.$$

$$= \lim_{x \to 0} \frac{(1+x^2) - (1-x^2)}{x^2\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} = \frac{2x^2}{x^2(\quad)}$$

$$= \frac{2x^2}{x^2\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)} = \lim_{x \to 0} \frac{2}{\sqrt{1+x^2} + \sqrt{1-x^2}} = \frac{2}{2} = 1$$

$$\frac{2}{2} = 1$$

TRIGONOMETRIC FUNCTION

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\sin x} = \lim\limits_{x\to 0}\dfrac{\tan x}{x} = \lim\limits_{x\to 0}\dfrac{x}{\tan x} = 1$

2. $\lim\limits_{x\to 0}\cos x = \lim\limits_{x\to 0}\sec x = 1$

3. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x} = \dfrac{\tan^{-1}x}{x} = 1$

4. $\lim\limits_{x\to\infty}\dfrac{\sin x}{x} = 0$

5. $\lim\limits_{x\to\infty}\dfrac{\sin\frac{1}{x}}{\frac{1}{x}} = 1$

$\left(\frac{1}{x}\to 0\right)$

Series

10. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$

11. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dots$

12. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dots$

13. $\tan x = x + \dfrac{x^3}{3} + \dfrac{2}{15}x^5 + \dots$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$$

$$\lim_{x\to 0}\frac{2\sin^2 x/2}{4\cdot\frac{x^2}{4}}$$

$$\frac{2}{4}\left(\frac{\sin(x/2)}{(x/2)}\right)^2$$

$$=\left(\frac{1}{2}\right)$$

9) Find the value of $\lim\limits_{x\to 0}\dfrac{\tan x - x - \dfrac{x^3}{3}}{x^5}$ $\left(\dfrac{0}{0}\right)$

$$\lim_{x\to 0}\frac{\tan x - x - \dfrac{x^3}{3}}{x^5} \quad \text{(Using exp. formula)}$$

$$= \lim_{x\to 0}\frac{\dfrac{2}{15}x^5(1+\text{terms containing higher power of x})}{x^5} = \frac{2}{15}$$

$$\frac{\left(\cancel{x}+\cancel{\dfrac{x^3}{3}}+\dfrac{2}{15}\cancel{x^5}\right)-\cancel{x}-\cancel{\dfrac{x^3}{3}}}{\cancel{x^5}}$$

$$= \frac{2}{15}$$

EXPONENTIAL & LOGARITHMIC FUNCTION

1. $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$ $\frac{e^x - 1}{x} = \log e$

2. $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = 1$ and $\lim_{x\to 0} \frac{\log_e(1-x)}{x} = -1$

3. $\lim_{n\to\infty} x^n = \begin{cases} 0, & 0 < x < 1 \\ 1, & x = 1 \\ \infty, & x > 1 \end{cases}$ $(.2)^\infty \longrightarrow 0$

series

4. $a^x = 1 + x\log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots$

5. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^9}{4!} \ldots$

6. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$ (if $|x| < 1$)

$\lim_{x\to 0} 2\left(\frac{2^{2x} - 1}{2x}\right) = 2\log 2$

$= \log 4$