Functions L-2

By
Ankush Garg(B.Tech, IIT Jodhpur)

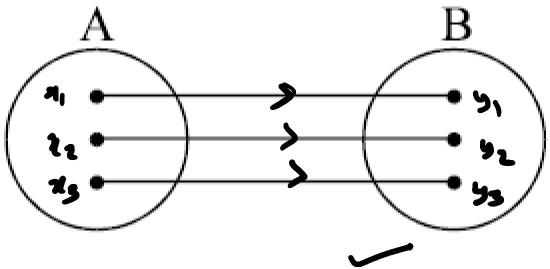
CLASSIFICATION OF FUNCTIONS :

One – One Function (Injective mapping) :

A function $f: A \rightarrow B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B .

Diagramatically an injective mapping can be shown as

$\boxed{x_1 \longrightarrow y_1}$



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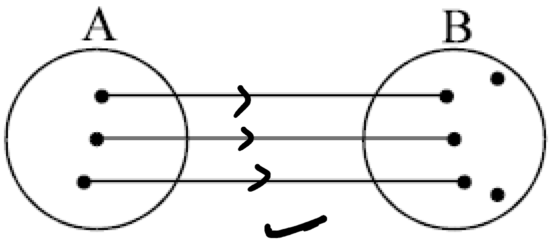
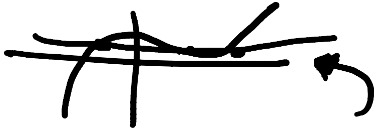
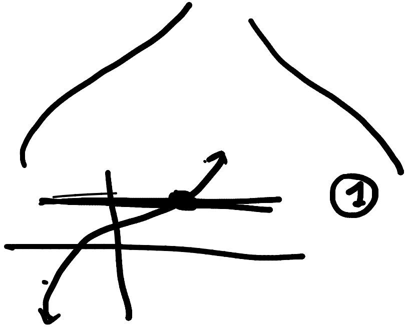


(i) Any function which is entirely increasing or decreasing in whole domain, then f(x) is one–one .

(ii) If any line parallel to x–axis cuts the graph of the function atmost at one point, then the function is one–one .

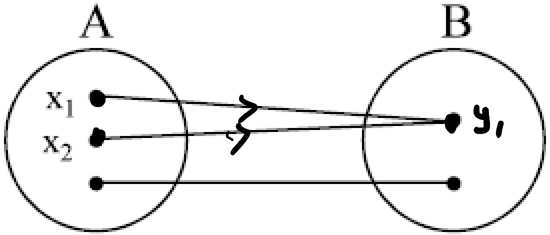




Many–one function :

A function $f : A \rightarrow B$ is said to be a many one function if two or more elements of A have the same f image in B .

Diagramatically a many one mapping can be shown as



Many - one

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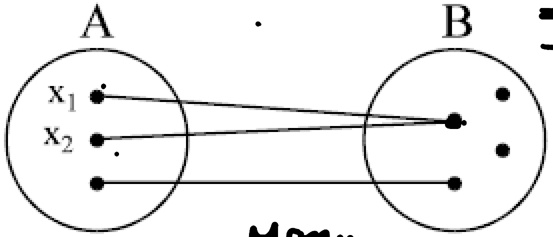
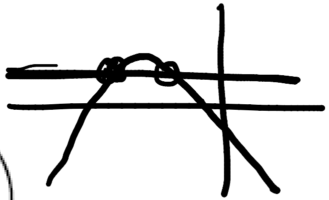
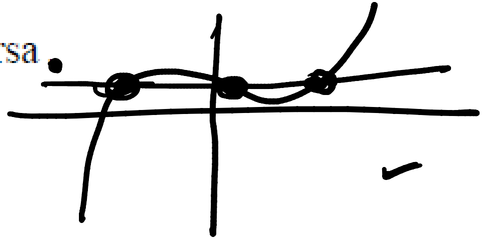


Many - one



(i) Any continuous function which has atleast one local maximum or local minimum, then $f(x)$ is many–one . In other words, if a line parallel to x–axis cuts the graph of the function atleast at two points, then f is many–one .

~~(ii) If a function~~ is one–one, it cannot be many–one and vice versa .

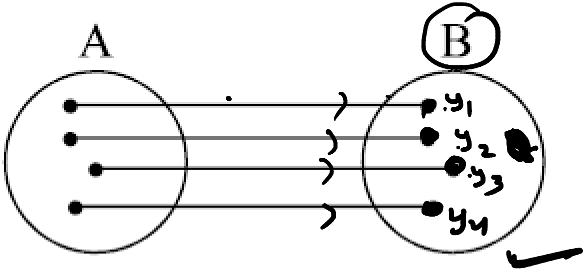


Many-one

Onto function (Surjective mapping) :

If the function $\boxed{f : A \to B}$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B .

Diagramatically surjective mapping can be shown as



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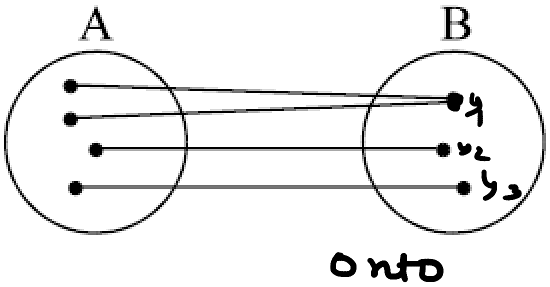
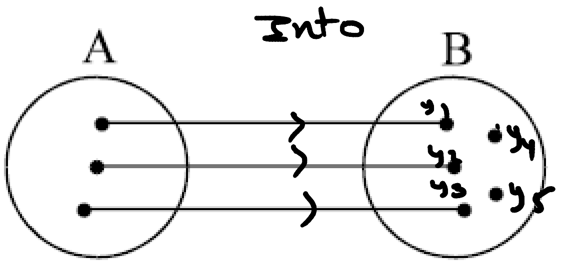


Note that : if $\boxed{\text{range} = \text{co–domain}}$, then f(x) is onto.

Into function : (Range $\neq$ co-Domain)

If $f: A \to B$ is such that there exists atleast one element in co–domain which is not the image of any element in domain, then $f(x)$ is into .

Diagramatically into function can be shown as



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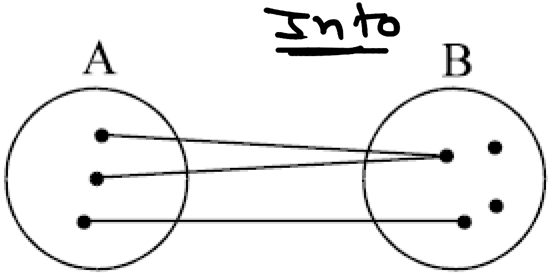


Note that : If a function is onto, it cannot be into and vice versa . A polynomial of degree even will always be into.

$\{y_1, y_2, y_3\}$

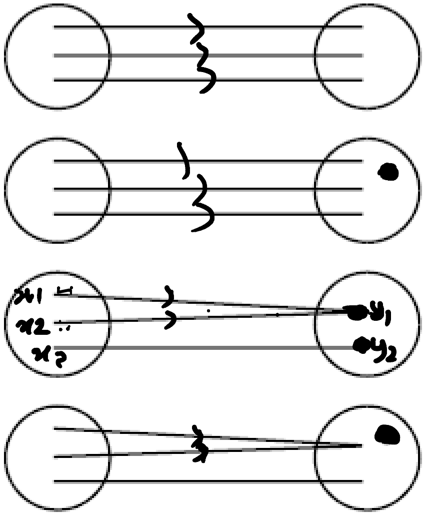
Thus a function can be one of these four types :

(a) one–one onto (injective & surjective)

(b) one–one into (injective but not surjective)

(c) many–one onto (surjective but not injective)

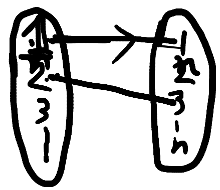
(d) many–one into (neither surjective nor injective)



$n(n-1)(n-2)$

$n!$

$A \rightarrow A$



$n \times n \times n$

$n^n \rightarrow$ functions

$\boxed{n!} \rightarrow$ one–one

(i) If f is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible functions.

Bijective functions

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \rightarrow A$ is n^n & out of it n ! are one one.

Problem

For real x, let $f(x) = x^3 + 5x + 1$, then

$f : A \to B$

(A) f is one-one but not onto R

(B) f is onto R but not one-one

(C) f is one-one and onto R

(D) f is neither one-one nor onto R

$$\boxed{f(x) = x^3 + 5x + 1}$$

$$f'(x) = 3x^2 + 5 > 0$$

one - one

f ↑ Increasing

$$\boxed{\text{Range} = R}$$

$(-\infty, \infty)$

Problem

The function $f : [2, \infty) \to Y$ defined by $f(x) = x^2 - 4x + 5$ is both one-one and onto if :

(A) $Y = R$ (B) $Y = [1, \infty)$ (C) $Y = [4, \infty)$ (D) $[5, \infty)$

9) If the function $f : \mathbf{R} - \{1, -1\} \to A$ defined by $f(x) = \dfrac{x^2}{1 - x^2}$, is surjective, then A is equal to

(2019 Main, 9 April I)

(a) $\mathbf{R} - \{-1\}$

(b) $[0, \infty)$

(c) $\mathbf{R} - [-1, 0)$

(d) $\mathbf{R} - (-1, 0)$

$R - [-1, 0)$

onto

Range = Co-Domain

= A

$$f(x) = \frac{x^2}{1-x^2}$$

$$y = \frac{x^2}{1-x^2}$$

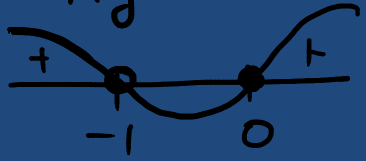
$$y - x^2 y = x^2$$

$$y = x^2(1+y)$$

$$x^2 = \frac{y}{1+y}$$

$$x = \pm\sqrt{\frac{y}{1+y}}$$

$$\frac{y}{1+y} \geq 0$$



$(-\infty, -1) \cup [0, \infty)$ Range

COMPOSITE FUNCTIONS

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x))$ $\forall\ x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow \boxed{g(f(x))}$

$f(x) = x^2$

$g(x) = \sin x$

$g(f(x)) = g(x^2)$

$= \sin(x^2)$

$fog = f(g(x))$ $\xrightarrow{x_1} \boxed{g(x)} \longrightarrow g(x_1)$

$= f(\sin x)$

$= (\sin x)^2$

f , g

bijective

$f(g(x))$

$g(f(x))$

bijective

$f(g(h)) = f(g(h(x))$

PROPERTIES OF COMPOSITE FUNCTIONS :

(i) The composite of functions is not commutative i.e. $gof \neq fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that $fo(goh)$ & $(fog)oh$ are defined, then $\boxed{fo(goh) = (fog)oh}$.

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.