JEE and NEET CRASH COURSE

Projectile Motion

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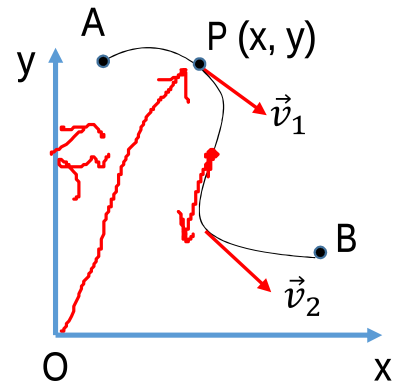
Motion in 2D or Motion in a Plane

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} = v_x\hat{\imath} + v_y\hat{\jmath}$$



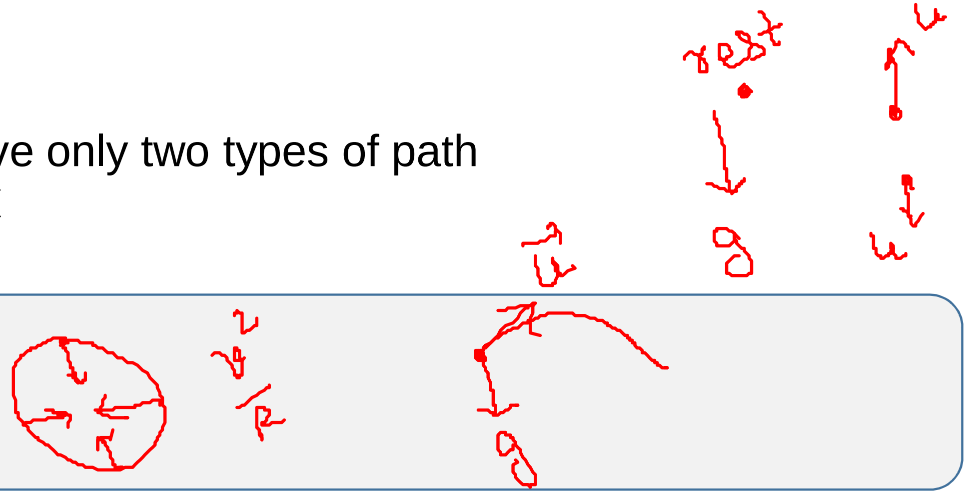
For any curved path, velocity is in the direction of tangent to the curve at that point.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{\imath} + \frac{dv_y}{dt}\hat{\jmath} = a_x\hat{\imath} + a_y\hat{\jmath}$$

Note:

If $\vec{a}$ = constant, then particle can have only two types of path

1. Straight Line, if $\vec{u} = 0 \ or \ \vec{u}\,||\vec{a}$
2. Parabolic, otherwise



Example

The P.V. for a particle is initially $\vec{r}_1 = -2\hat{i} - 2\hat{j}$ ***and then later is*** $\vec{r}_2 = 8\hat{i} + 10\hat{j}$. ***What is the displacement?***

$$\vec{S} = \vec{r}_2 - \vec{r}_1 = (8+2)\hat{i} + (10+2)\hat{j}$$

$$= 10\hat{i} + 12\hat{j}$$

Example

The equations of motion of a projectile thrown in x–y plane from origin are $x = 8t$, $y = 6t - 10t^2$ then the angle of projectile is –

(A) $\tan^{-1}(3/4)$ (B) $\tan^{-1}(4/3)$ (C) $\sin^{-1}(3/4)$ (D) $\cos^{-1}(3/4)$

$$v_x = \frac{dx}{dx} = 8$$

$$v_y = \frac{dy}{dt} = 6 - 20t$$

At $t = 0$;

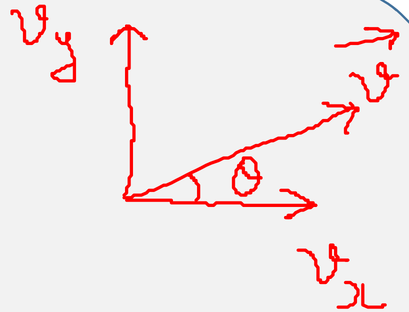
$v_x = 8$

$v_y = 6$

$$\tan\theta = \frac{v_y}{v_x}$$

$$= \frac{6}{8} = \frac{3}{4}$$

$$\theta = \tan^{-1}(3/4)$$



$$\tan\theta = \frac{v_y}{v_x}$$

Example

A particle is moving eastward with a speed of 5 m/s. After 10 seconds, the direction changes towards north, but speed remains same. The average acceleration in this time is

(A) zero

(B) $\frac{1}{\sqrt{2}}$ m/s^2 towards N-W

(C) $\frac{1}{\sqrt{2}}$ m/s^2 towards N-E

(D) $\frac{1}{\sqrt{2}}$ m/s^2 towards S-W

$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$

$\theta = 90^\circ$

$R = \sqrt{A^2 + B^2}$

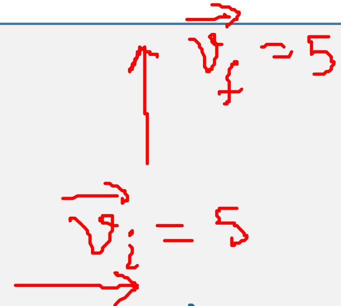
If $A = B$

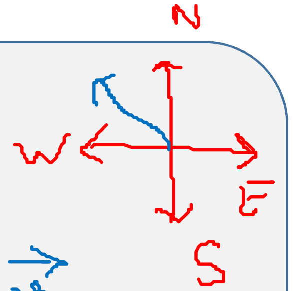
$R = \sqrt{2}A$

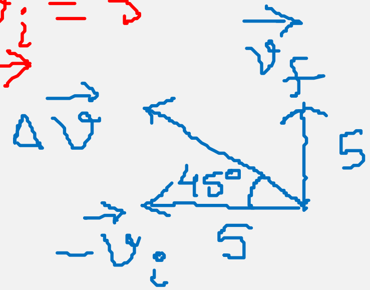
$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{t_f - t_i} = \frac{\vec{v}_f + (-\vec{v}_i)}{10}$$

$$= \frac{5\sqrt{2}}{10} = \frac{1}{\sqrt{2}}\ m/s^2$$

N-W







Example

A particle moves in space along the path $z = ax^3 + by^2$ in such a way that $\frac{dx}{dt} = c = \frac{dy}{dt}$ where a, b and c are constants. The acceleration of the particle is:

(a) $(6ac^2x + 2bc^2)\hat{k}$

(b) $(2ax^2 + 6by^2)\hat{k}$

(c) $(4bc^2x + 3ac^2)\hat{k}$

(d) $(bc^2x + 2by)\hat{k}$

$$v_x = \frac{dx}{dt} = c$$

$$v_y = \frac{dy}{dt} = c$$

$$a_x = \frac{dv_x}{dt} = 0$$

$$a_y = \frac{dv_y}{dt} = 0$$

$$z = ax^3 + by^2$$

$$v_z = \frac{dz}{dt} = 3ax^2\frac{dx}{dt} + 2by\frac{dy}{dt}$$

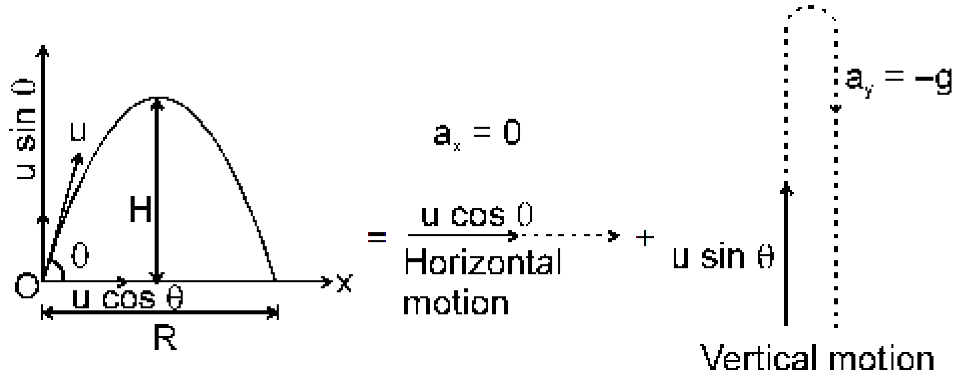
$$= 3acx^2 + 2bcy$$

$$a_z = \frac{dv_z}{dt} = 6acx\frac{dx}{dt} + 2bc\frac{dy}{dt}$$

$$= (6ac^2x + 2bc^2)\hat{k}$$

PROJECTILE MOTION

It is two dimensional motion with constant acceleration



	Horizontal direction	Vertical direction
(a)	Initial velocity $u_x = u\cos\theta$	Initial velocity $u_y = u\sin\theta$
(b)	Acceleration $a_x = 0$	Acceleration $a_y = -g$
(c)	Velocity after time t, $v_x = u\cos\theta$	Velocity after time t, $v_y = u\sin\theta - gt$
(d)	Displacement, $x = u\cos\theta t$	Displacement, $y = u\sin\theta t - \frac{1}{2}gt^2$

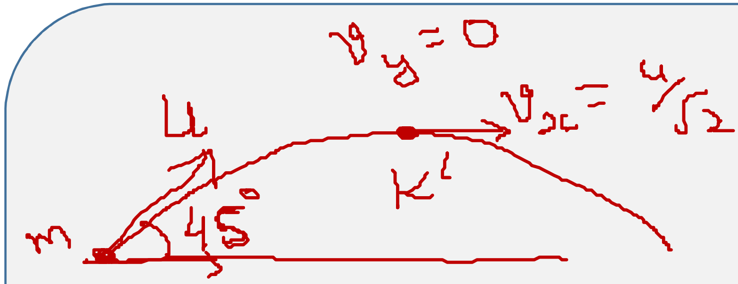
$v = u + at$

$s = ut + \frac{1}{2}at^2$

Example

A ball is projected with kinetic energy K at an angle of 45° to the horizontal. At the highest point during its flight, its kinetic energy will be

(A) K **(B)** $\frac{K}{\sqrt{2}}$ ✓**(C)** $\frac{K}{2}$ **(D)** zero



$u_x = u\cos 45^\circ$

$= u/\sqrt{2}$

$K = \frac{1}{2}mu^2$

$K' = \frac{1}{2}m\left(u/\sqrt{2}\right)^2$

$= \frac{1}{2}\left(m \cdot \frac{u^2}{2}\right) = \frac{K}{2}$

Time of Flight, Range and Max. Height

Time of flight :

The displacement along vertical direction is zero for the complete flight.

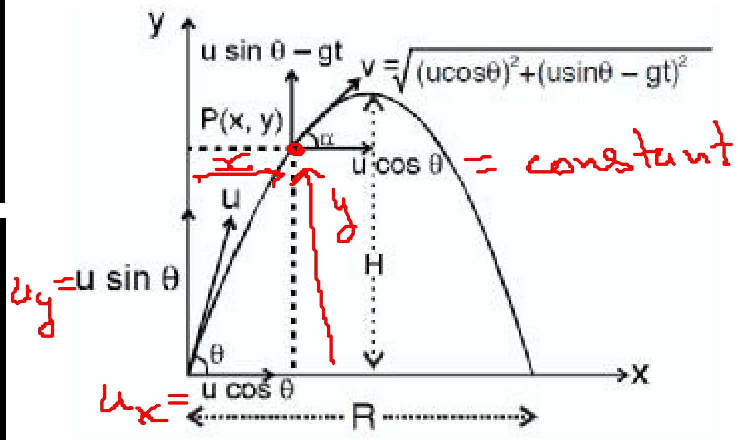
Hence, along vertical direction net displacement = 0

$$\Rightarrow \quad (u\sin\theta)T - \frac{1}{2}gT^2 = 0 \quad \Rightarrow \quad T = \frac{2u\sin\theta}{g} = \frac{2u_y}{g}$$

Horizontal range :

$$R = u_x . T \quad > \quad R = u\cos\theta . \frac{2u\sin\theta}{g}$$

$$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u_x u_y}{g}$$



Maximum height :

At the highest point of its trajectory, particle moves horizontally, and hence vertical component of velocity is zero.

Using 3rd equation of motion i.e. $v^2 = u^2 + 2as$

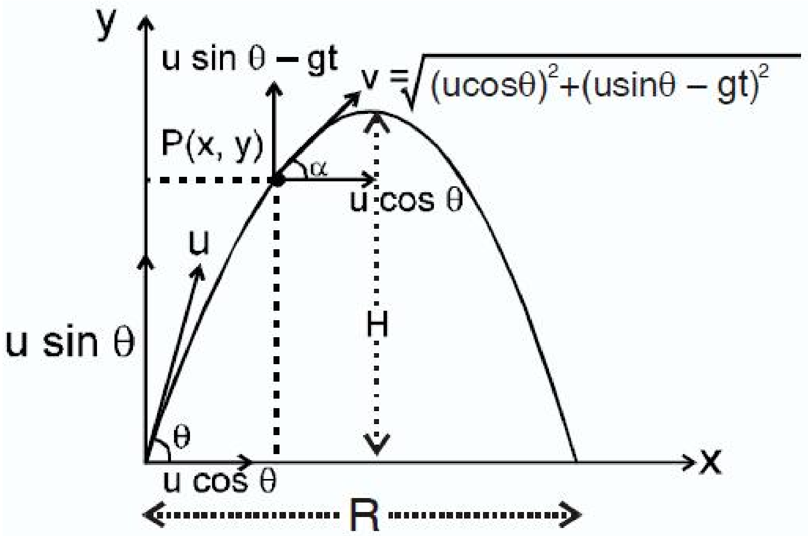
we have for vertical direction $0 = u^2\sin^2\theta - 2gH \quad \Rightarrow \quad H = \frac{u^2\sin^2\theta}{2g} = \frac{u_y^2}{2g}$

Resultant Velocity at time t

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \quad = u\cos\theta\ \hat{i} + (u\sin\theta - gt)\ \hat{j}$$

Where, $|\vec{v}| = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$ and $\tan\alpha = v_y / v_x$

Also, $v\cos\alpha = u\cos\theta \Rightarrow v = \dfrac{u\cos\theta}{\cos\alpha}$



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Example

A particle is projected at an angle of 60° above the horizontal with a speed of 10 m/s. After some time the direction of its velocity makes an angle of 30° above the horizontal. The speed of the particle at this instant is:

(a) $\frac{5}{\sqrt{3}}$ m/s (b) $5\sqrt{3}$ m/s (c) 5 m/s (d) $\frac{10}{\sqrt{3}}$ m/s

$$v \cos 30^\circ = u \cos 60^\circ$$

$$v \frac{\sqrt{3}}{\cancel{2}} = 10 \times \frac{1}{\cancel{2}}$$

$$\boxed{v = \frac{10}{\sqrt{3}}}$$

$\sin(180-\theta) = \sin\theta$

Important Results

$R = \dfrac{u^2 \sin 2\theta}{g}$

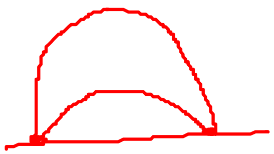
$H = \dfrac{u^2 \sin^2\theta}{2g}$

For H_{max} ; $\theta = 90^\circ$

$H_{max} = \dfrac{u^2}{2g}$

- For maximum range $\theta = 45^\circ$

  $R_{max} = \dfrac{u^2}{g} \Rightarrow H_{max} = \dfrac{R_{max}}{2}$



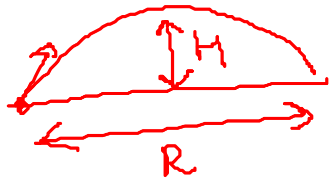
- We get the same range for two angle of projections α and $(90 - \alpha)$ but in both cases, maximum heights attained by the particles are different.

  This is because, $R = \dfrac{u^2 \sin 2\theta}{g}$, and $\sin 2(90 - \alpha) = \sin 180 - 2\alpha = \sin 2\alpha$

- If $R = H$

  i.e. $\dfrac{u^2 \sin 2\theta}{g} = \dfrac{u^2 \sin^2\theta}{2g}$

  $\Rightarrow \tan\theta = 4$



$T = \dfrac{2u_y}{g}$

$H = \dfrac{u_y^2}{2g}$

- Range can also be expressed as

  $R = \dfrac{u^2 \sin 2\theta}{g} = \dfrac{2u \sin\theta . u \cos\theta}{g} = \dfrac{2u_x u_y}{g}$

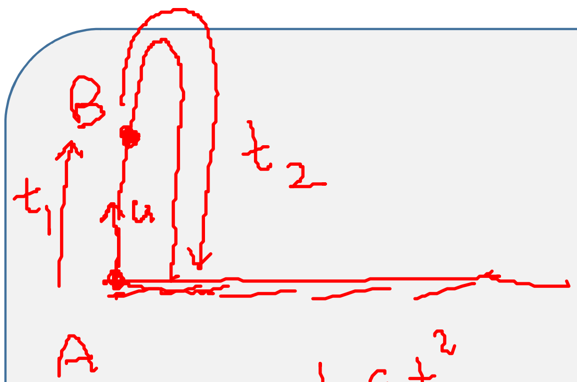
Example

A particle is projected vertically upwards from a points A on the ground. It takes time t_1 to reach a point B, but it still continues to move up. If it takes further t_2 time to reach the ground from point B. Then height of point B from the ground is:

(a) $\frac{1}{2}g(t_1 + t_2)^2$ (b) $g\, t_1\, t_2$ (c) $\frac{1}{8}g\,(t_1 + t_2)^2$ (d) $\frac{1}{2}g\, t_1\, t_2$

$T=\frac{2u_y}{g}$

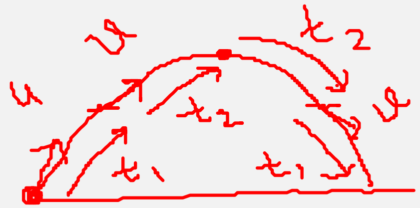
$H=\frac{u_y^2}{2g}$



$T=t_1+t_2$

$\frac{2u}{g}=t_1+t_2$

$u=\frac{g}{2}(t_1+t_2)$



$s=ut+\frac{1}{2}at^2$

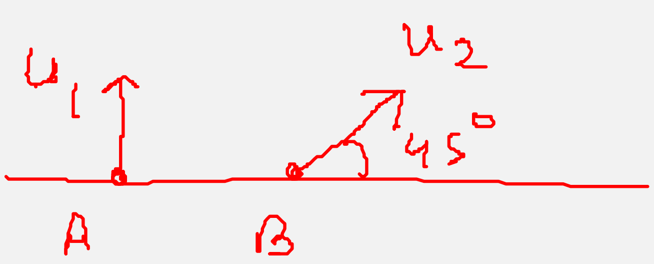
$h=\frac{g}{2}(t_1+t_2)t_1-\frac{1}{2}gt_1^2$

$=\frac{g}{2}\left(\cancel{t_1^2}+t_1t_2-\cancel{t_1^2}\right)=\frac{g t_1 t_2}{2}$

Example

A particle A is projected vertically upwards. Another particle B of same mass is projected at an angle of 45°. Both reach the same height. The ratio of the initial kinetic energy of A to that of B is:

(a) 1 : 2 ✓ (b) 2 : 1 (c) 1 : $\sqrt{2}$ (d) $\sqrt{2}$: 1



$$H_A = H_B$$

$$\frac{u_1^2}{\cancel{2g}} = \frac{u_2^2 \sin^2 45^\circ}{\cancel{2g}}$$

$$u_1 = u_2 \sin 45^\circ$$

$$u_1 = u_2/\sqrt{2} \Rightarrow \frac{u_1}{u_2} = \frac{1}{\sqrt{2}}$$

$$\frac{K_A}{K_B} = \frac{\cancel{\frac{1}{2}} \cancel{m} u_1^2}{\cancel{\frac{1}{2}} \cancel{m} u_2^2} = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

Equation of Trajectory

The path followed by a particle (here projectile) during its motion is called its **Trajectory**. Equation of trajectory is the relation between instantaneous coordinates (Here x & y coordinate) of the particle.

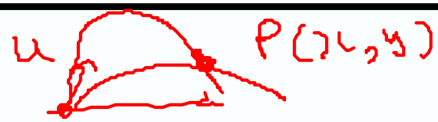
If we consider the horizontal direction, $x = u_x . t = u \cos\theta . t$ ...(1)

For vertical direction : $y = u_y . t - 1/2\, gt^2 = u \sin\theta . t - 1/2\, gt^2$ ...(2)

Eliminating 't' from equation (1) & (2)

$$y = u \sin\theta . \frac{x}{u\cos\theta} - \frac{1}{2} g \left(\frac{x}{u\cos\theta}\right)^2 \Rightarrow \boxed{\mathbf{y = x \tan\theta - \frac{gx^2}{2u^2\cos^2\theta}}}$$

Other forms of trajectory equation :



- $$\mathbf{y = x \tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}}$$
- $$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \Rightarrow y = x\tan\theta\left[1 - \frac{gx^2}{2u^2\cos^2\theta\tan\theta}\right]$$

$$\Rightarrow y = x\tan\theta\left[1 - \frac{gx}{2u^2\sin\theta\cos\theta}\right] \Rightarrow \boxed{\mathbf{y = x\tan\theta\left[1 - \frac{x}{R}\right]}}$$

Example

A projectile is given an initial velocity of $\hat{i} + 2\hat{j}$. The Cartesian equation of its path is: (g = 10 m/s^2)

(a) $y = 2x - 5x^2$ (b) $y = x - 5x^2$ (c) $4y = 2x - 5x^2$ (d) $y = 2x - 25x^2$

$u_x = 1$ $a_x = 0$

$u_y = 2$ $a_y = -10$

$x = u_x t = t$

$y = u_y t + \frac{1}{2} a_y t^2$

$= 2t - 5t^2$

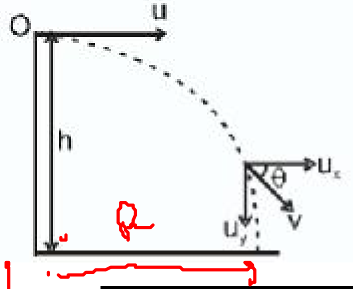
$y = 2x - 5x^2$

$u = \sqrt{u_x^2 + u_y^2}$

$\tan\theta = \frac{u_y}{u_x} = 2$

Projectile thrown horizontally from some Height

Consider a projectile thrown from point O at some height h from the ground with a velocity u. Now we shall study the characteristics of projectile motion by resolving the motion along horizontal and vertical directions.



	Horizontal direction	Vertical direction
(i)	Initial velocity $u_x = u$	Initial velocity $u_y = 0$
(ii)	Acceleration $a_x = 0$	Acceleration $a_y = g$ (downward)

$h = \frac{1}{2} g T^2$

$R = u_x T$

$v_x = u$

$v_y = \sqrt{2gH}$

Time of flight : $t = \sqrt{\frac{2h}{g}}$

Horizontal range : $R = u\sqrt{\frac{2h}{g}}$

Velocity with which the projectile hits the ground : $V = \sqrt{u^2 + 2gh}$

$v_y^2 = u_y^2 + 2a_y y$

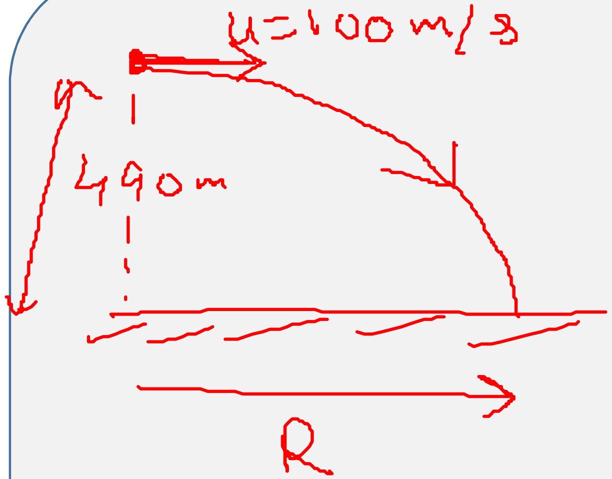
$= 0 + 2gH$

Example

= 100 m/s

An aeroplane flying horizontally with a speed of 360 km/h releases a bomb at a height of 490 m from the ground. When will the bomb strike the bomb strike the ground?

(A) 8s ***(B) 6s*** ***(C) 7s*** ***(D) 10s.***



$$T = \sqrt{\frac{2H}{g}} = \sqrt{\frac{\cancel{2}\times 490}{\cancel{9.8}\ 4.9}} = \sqrt{100}$$

$$= 10\ s$$

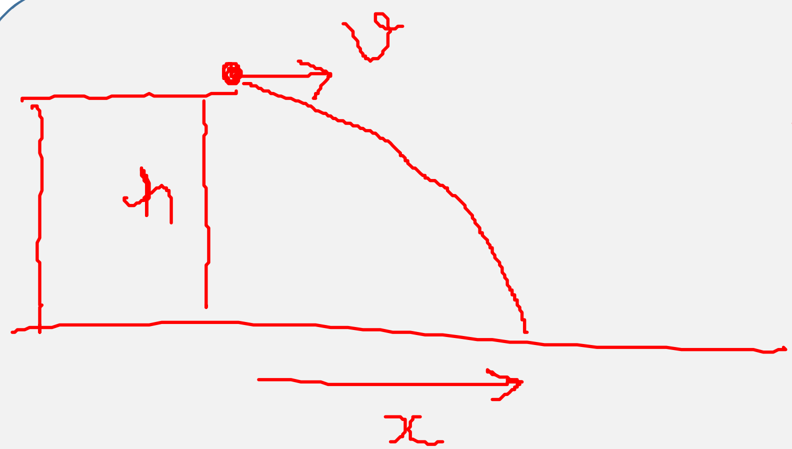
$$R = uT = 1000\ m$$

$$= 1\ Km$$

Example

A glass marble projected horizontally from the top of a table falls at a distance x from the edge of the table. If h is the ~~highest~~ height of the table, then the velocity of projection is

(A) $h\sqrt{\frac{g}{2x}}$ (B) $x\sqrt{\frac{g}{2h}}$ (C) gxh (D) $gx + h$.



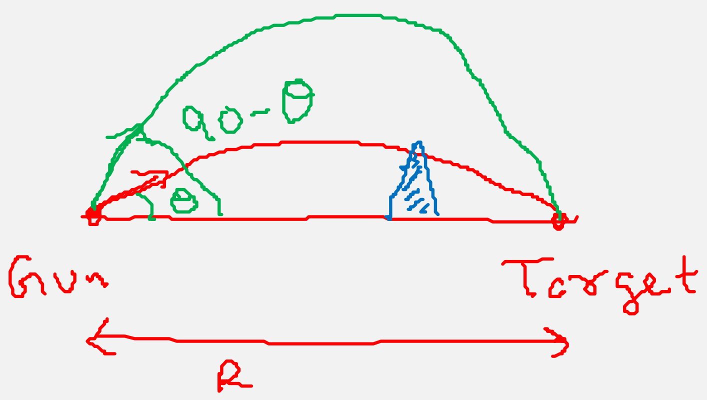
$$x = v\sqrt{\frac{2h}{g}}$$

$$v = x\sqrt{\frac{g}{2h}}$$

Example

It was calculated that a shell when fired from a gun with a certain velocity and at an angle of elevation $\frac{5\pi}{36}$ radians should strike a given target. In actual practice it was found that a hill just prevented in the trajectory. At what angle of elevation should the gun be fired to hit the target.

(A) $\frac{5\pi}{36}$ radians (B) $\frac{11\pi}{36}$ radians (C) $\frac{7\pi}{36}$ radians (D) $\frac{13\pi}{36}$ radians



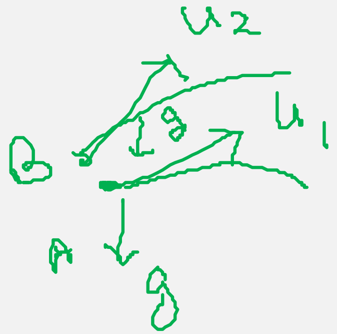
$$\frac{\pi}{2} - \frac{5\pi}{36}$$

$$= \frac{18\pi - 5\pi}{36} = \frac{13\pi}{36}$$

Example

The locus of a projectile relative to another projectile is a

(A) straight line ***(B) circle*** ***(C) ellipse*** ***(D) parabola***



$\vec{a}_{rel} = 0$

If $\vec{a}_{rel} = 0$

then $\vec{v}_{rel} =$ const

$\vec{v} =$ const