Straight Lines

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POSITION OF A POINT W.R.T. A LINE :

If the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lies on the same side of the line $Ax + Bx + C$ then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of the same sign and if (x_1, y_1) and (x_2, y_2) are on the opposite side of the line then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of opposite sign.

The position of the point (8,–9) with respect to the lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ is -

(A) point lies on the same side of the lines

(B) point lies on one of the lines

(C) point lies on the different sides of the line

(D) point lies between the lines

$6x+9y+8=0$

$6x+9y=-8$

$\frac{x}{-8/6} + \frac{y}{-8/9} = 1$

$2x+3y=4.$

$\frac{x}{2} + \frac{y}{4/3} = 1$

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LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

The length of perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

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ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES :

If m_1 & m_2 are the slopes of two intersecting straight lines $(m_1 m_2 \neq -1)$ & θ is the acute angle between them, then $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$.

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

→ 2 lines are parallel $\Rightarrow \tan\theta = \tan 0^\circ = 0 \Rightarrow \dfrac{m_1 - m_2}{1 + m_1 m_2} = 0 \Rightarrow m_1 = m_2$

→ 2 lines are ⊥ $\Rightarrow \theta = 90^\circ \Rightarrow \tan\theta = \infty \Rightarrow \dfrac{m_1 - m_2}{1 + m_1 m_2} = \dfrac{1}{0}$

$\boxed{m_1 m_2 = -1}$

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Problems

A line L passes through the points (1, 1) and (0, 2) and another line M which is perpendicular to L passes through the point (0, −1/2).The area of the triangle formed by these lines with y-axis is -

(A) 25/8 (B) 25/16 (C) 25/4 (D) 25/32

$y-1 = \frac{2-1}{0-1}(x-1)$

$y-1 = -1(x-1)$

$\boxed{x+y=2}$ L $m_L = -1$

$\Rightarrow m_M = 1$

eqn of M $\Rightarrow$ $y+\frac{1}{2} = 1(x-0)$

~~2y+1=2x~~ $\boxed{x-y=\frac{1}{2}}$ M.

$2x = 5/2$

$x = 5/4.$

$y = 2 - \frac{5}{4} = 3/4.$

Area of A

$= \frac{1}{2} \times \frac{5}{2} \times \frac{5}{4}.$

$= 25/16$

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Problems

$ax+by+c=0$

$m=-a/b$

If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then the value of k is -

(A) 1 (B) 2 (C) 3 (D) 4

$(3-k)x+(4-k)y+5-3k=0$

$m=-\frac{(3-k)}{4-k}=\frac{k-3}{4-k}$

$\frac{k-3}{4-k}=\frac{1}{0} \Rightarrow 4-k=0$

$\Rightarrow k=4$

$m=\tan 90^\circ=\infty=1/0$

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PARALLEL LINES :

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$. Where k is a parameter.

(ii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ & $ax + by + c_2 = 0$ is

$$\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right| .$$

Note that the coefficients of x & y in both the equations must be same.

$ax + by + c = 0$

$\boxed{ax + by + k = 0}$

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PERPENDICULAR LINES :

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.

(ii) Straight lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ are at right angles if & only if $aa' + bb' = 0$.

$ax + by + c = 0$

$\perp$ line : $bx - ay + k = 0$

$\perp$

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Problems

Lines are drawn parallel to the line $4x - 3y + 2 = 0$, at a distance $\frac{3}{5}$ from the origin. Then which one of the following points lies on any of these lines?

(2019 Main, 10 April I)

(a) $\left(-\frac{1}{4}, -\frac{2}{3}\right)$ (b) $\left(-\frac{1}{4}, \frac{2}{3}\right)$ 4

(c) $\left(\frac{1}{4}, -\frac{1}{3}\right)$ (d) $\left(\frac{1}{4}, \frac{1}{3}\right)$

$4x-3y+2=0 \Rightarrow 4x-3y=-2$

$\boxed{\frac{x}{-2} + \frac{y}{2/3} = 1}$

$4x-3y+k=0$

$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

Line 1: $4x-3y+3=0$ — (1)

Line 2: $4x-3y-3=0$ — (2)

length of $\perp$ from 0,0 $= \left|\frac{4\times0-3\times0+k}{\sqrt{4^2+3^2}}\right| = \frac{|k|}{5} = \frac{3}{5}$

$|k| = 3 \Rightarrow k = \pm 3$

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Problems

If the straight line, $2x - 3y + 17 = 0$ is perpendicular to the line passing through the points (7, 17) and (15, β), then β equals **(2019 Main, 12 Jan I)**

(a) $\frac{35}{3}$ (b) -5 (c) $-\frac{35}{3}$ (d) 5

$m_1 \times m_2 = -1$

$\frac{2}{3} \times \frac{\beta - 17}{8} = -1$

$\beta - 17 = -12$

$\beta = 17 - 12$

$\beta = 5$

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Problems

Suppose that the points (h, k), $(1, 2)$ and $(-3, 4)$ lie on the line L_1. If a line L_2 passing through the points (h, k) and $(4, 3)$ is perpendicular to L_1, then k/h equals

(2019 Main, 8 April II)

(a) $-\frac{1}{7}$ (b) $\frac{1}{3}$ $1/3$

(c) 3 (d) 0

eqn of L_1 = $y-4 = -\frac{1}{2}(x+3)$

$2y-8 = -x-3$

$\boxed{x+2y=5}$ — L_1

$m_{L_1} = \frac{4-2}{-3-1} = \frac{2}{-4} = -\frac{1}{2}$

$m_{L_2} = 2$

eqn of L_2 $2x-y+k=0$ $(4,3)$

$8-3+k=0 \Rightarrow k=-5$

$\boxed{2x-y=5}$ — L_2

$2x+4y=10$

$2x-y=5$

$5y=5$

$y=1$ $x=3$

$\cancel{2x=y+5}$

$\cancel{2x=\frac{17}{5}+5=\frac{42}{5}}$

$\cancel{x=21/5}$

$h=21/5$, $k=17/5$

$k/h = 17/21$

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CONCURRENCY OF THREE LINES :

Three lines are said to be concurrent if they pass through a common point, i.e. they meet at a point. Thus, if three line are concurrent the point of intersection of two lines lies on the third line. Let the three concurrent lines be

$$a_1x + b_1y + c_1 = 0 \quad \ldots(i)$$
$$a_2x + b_2y + c_2 = 0 \quad \ldots(ii)$$
$$a_3x + b_3y + c_3 = 0 \quad \ldots(iii)$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

Problems

Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + \lambda = 0$ are concurrent.

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

Find λ

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FAMILY OF STRAIGHT LINES :

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$

Then, the general equation of any straight line passing through the point of intersection of lines L_1 and L_2 is given by $\boxed{L_1 + \lambda L_2 = 0}$, where $\lambda \in R$

parameter

These lines form a family of straight line

Also this general equation satisfies point of intersection of L_1 and L_2 for any value of λ.

Conversely, if a variable line is expressed in the form of $L_1 + \lambda L_2 = 0$ $(\lambda \in R)$ then it always passes through fixed point which is the point of intersection of $L_1 = 0$ and $L_2 = 0$.

pass through p.o.i of L_1 & L_2

$\Rightarrow (2-\lambda)x + (3+\lambda)y + 5 - 2\lambda = 0 \quad \Rightarrow (2x+3y+5) + \lambda(-x+y-2) = 0$

$L_1 + \lambda L_2 = 0$

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Problems

Find the equation of the line through the point of intersection of the lines $3x - 4y + 1 = 0$ & $5x + y - 1 = 0$ and perpendicular to the line $2x - 3y = 10$.

$L_1 + \lambda L_2 = 0$

$3x - 4y + 1 + \lambda(5x + y - 1) = 0$

$(3+5\lambda)x + (\lambda - 4)y + 1 - \lambda = 0$

$m = -\frac{(3+5\lambda)}{\lambda - 4} = \frac{3+5\lambda}{4-\lambda}$

$m = \frac{2}{3}$

$\perp$

$m_1 m_2 = -1$

$\frac{3+5\lambda}{4-\lambda} \times \frac{2}{3} = -1$

$6 + 10\lambda = -12 + 3\lambda$

$7\lambda = -18$

$\lambda = -18/7$

$\lambda = -18/7$

$\left(3 - \frac{90}{7}\right)x + \left(-\frac{18}{7} - 4\right)y + 1 + \frac{18}{7} = 0$

$\boxed{-69x - 46y + 25 = 0}$

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Problems

$L_1 + \lambda L_2 = 0$

The line $(p + 2q)x + (p - 3q)y = p - q$ for different values of p and q passes through a fixed point whose co-ordinates are -

(A) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (B) $\left(\frac{2}{5}, \frac{2}{5}\right)$ (C) $\left(\frac{3}{5}, \frac{3}{5}\right)$ (D) $\left(\frac{2}{5}, \frac{3}{5}\right)$

$px + 2qx + py - 3qy = p - q$

$p(x + y - 1) + q(2x - 3y + 1) = 0$

$x + y - 1 + \frac{q}{p}(2x - 3y + 1) = 0$

$\boxed{x + y - 1 + \lambda(2x - 3y + 1) = 0}$

L_1 L_2

$2x - 3y + 1 = 0$

$2x + 2y - 2 = 0$

$-5y + 3 = 0$

$y = 3/5$

$x + y = 1$

$x + \frac{3}{5} = 1$

$x = 2/5$

$\left(2/5, \frac{3}{5}\right)$

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Problems

Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true? **(2019 Main, 9 Jan I)**

(a) Each line passes through the origin.

(b) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$

(c) The lines are all parallel

(d) The lines are not concurrent

$px + qy + r = 0$

$\left(\frac{3}{4}, \frac{2}{4}\right)$ always satisfies

$\left(\frac{3}{4}, \frac{1}{2}\right)$

$3p + 2q + 4r = 0$

$\left(\frac{3}{4}\right)p + \left(\frac{2}{4}\right)q + r = 0$ ✓ Always true

$(1, -2)$

$ax + by + c = 0$ variable line.

a, b, c are in A.P then line always passes through

$2b = a + c$

$a - 2b + c = 0$

$a \times 1 + b(-2) + c = 0$

$\left(\frac{3}{4}, \frac{1}{2}\right)$

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Optics Based Problem :

(i) Image of a point in a line.

(ii) A ray of light incident along the line $lx + my + n = 0$ and strikes a line mirror. if the equation of normal on the line mirror at the point of incidence is $px + qy + r = 0$ then find the equation of the reflected ray

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REFLECTION OF A POINT :

Let P(x, y) be any point, then its image with respect to

(a) x-axis is Q(x, –y)

(b) y-axis is R(–x, y)

(c) origin is S(–x,–y)

(d) line y = x is T(y, x)

(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line ax + by + c = 0 is given by following formula.

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line ax + by + c = 0 is given by following formula.

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

$T(x_1, y_1)$

(x, y)

Foot of ⊥

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = -\left(\frac{ax_1 + by_1 + c}{a^2 + b^2}\right)$$

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and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line ax + by + c = 0 is given by following formula.

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$

Find mirror image of 2,-3 in y=x

(-3,2) Ans.

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Problems

The point (4, 1) undergoes the following transformations, then the match the correct alternatives :

	Column-I		Column-II
(A)	Reflection about x-axis is	(p)	(4, –1)
(B)	Reflection about y-axis is	(q)	(–4, –1)
(C)	Reflection about origin is	(r)	$\left(-\frac{12}{25}, -\frac{59}{25}\right)$
(D)	Reflection about the line y = x is	(s)	(–4, 1)
(E)	Reflection about the line 4x + 3y – 5 = 0 is	(t)	(1, 4)

$$\frac{x-4}{4} = \frac{y-1}{3} = -2\left(\frac{16+3-5}{4^2+3^2}\right)$$

$$\frac{x-4}{4} = \frac{y-1}{3} = -2 \times \frac{14}{25} \Rightarrow \frac{x-4}{4} = \frac{-28}{25}$$

$$x-4 = \frac{-112}{25} \Rightarrow x = 4 - \frac{112}{25} = \frac{-12}{25}$$

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Optics Based Problem :

(i) Image of a point in a line.

(ii) A ray of light incident along the line $lx + my + n = 0$ and strikes a line mirror. if the equation of normal on the line mirror at the point of incidence is $px + qy + r = 0$ then find the equation of the reflected ray

Problems

A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

$x - 2y - 3 = 0$

$3x - 2y - 5 = 0$

$2x - 2 = 0$

$x = 1, \; y = -1$

→ Find image of point P(3,0) in line mirror

→ Apply 2 point form to get eqn of reflected ray

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BISECTORS OF THE ANGLES BETWEEN TWO LINES :

Equations of the bisectors of angles between the lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ $(ab' \neq a'b)$ are : $\dfrac{ax + by + c}{\sqrt{a^2 + b^2}} = \pm \dfrac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$

bisectors of any given lines are always ⊥.

$$\left| \frac{ax+by+c}{\sqrt{a^2+b^2}} \right| = \left| \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}} \right|$$

→ Find Acute angle bisector

→ Find Obtuse angle bisector

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To discriminate between the acute angle bisector & the obtuse angle bisector

To discriminate between acute angle bisector & obtuse angle bisector proceed as follows Write $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that constant terms are positive.
If $aa' + bb' < 0$, then the angle between the lines that contains the origin is acute and the equation of the bisector of this acute angle is $\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$

If, however, $aa' + bb' > 0$, then the angle between the lines that contains the origin is obtuse & the equation of the bisector of this obtuse angle is:

$$\frac{ax + by + c}{\sqrt{a^2 + b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}} \; ; \text{ therefore } \frac{ax + by + c}{\sqrt{a^2 + b^2}} = - \frac{a'x + b'y + c'}{\sqrt{a'^2 + b'^2}}$$

is the equation of other bisector.

Obtuse angle bisector will be of same sign as of $aa' + bb'$ when c & c' are +ve.

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Problems

For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the

(i) bisector of the obtuse angle between them.

(ii) bisector of the acute angle between them.

(1) Make c_1, c_2 +ve

(2) $a_1a_2 + b_1b_2$

$-4x - 3y + 6 = 0$

$5x + 12y + 9 = 0$

$a_1a_2 + b_1b_2$

$(-4)(5) + (-3)(12) < 0$

– ve obtuse.

+ ve acute

$$\boxed{\frac{-4x-3y+6}{5} = \pm \frac{5x+12y+9}{13}}$$

(+) $-52x - 39y + 78 = 25x + 60y + 45$

$77x + 99y = 33$

$\boxed{7x + 9y = 3}$ (acute)

(–) $-52x - 39y + 78 = -25x - 60y - 45$

$27x - 21y \mp 123 = 0$

$\boxed{9x - 7y \mp 41 = 0}$ (obtuse)