Straight Lines

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Concepts

DISTANCE FORMULA :

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

Problems

The number of points on x-axis which are at a distance c(c < 3) from the point (2, 3) is

(A) 2 (B) 1 (C) infinite (D) no point

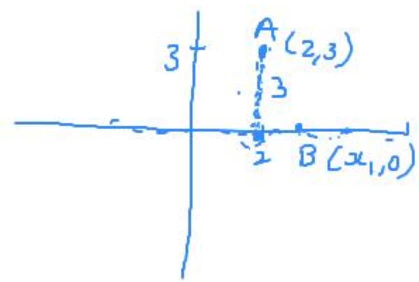
AB < 3

$$\sqrt{(x_1-2)^2+3^2} < 3$$

$$(x_1-2)^2+9 < 9$$

$$(x_1-2)^2 < 0$$

Not possible



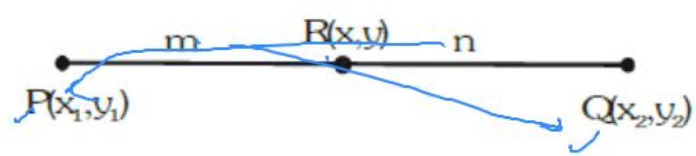
Concepts

SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in the ratio m:n is given by :

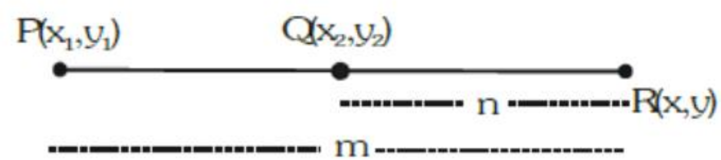
(a) For internal division : P - R - Q $\Rightarrow$ R divides line segment PQ, internally.

$$(x,\ y) \equiv \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$



(b) For external division : R - P - Q or P - Q - R $\Rightarrow$ R divides line segment PQ, externally.

$$(x,\ y) \equiv \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

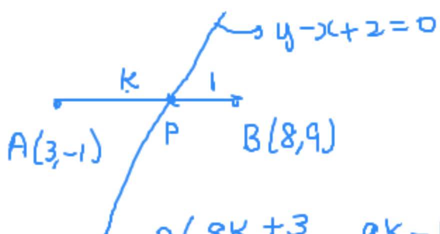


$\frac{(PR)}{(QR)} < 1 \Rightarrow$ R lies on the left of P & $\frac{(PR)}{(QR)} > 1 \Rightarrow$ R lies on the right of Q

Note : If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Problems

Determine the ratio in which $y - x + 2 = 0$ divides the line joining $(3, -1)$ and $(8, 9)$.



$$P\left(\frac{8k+3}{k+1}, \frac{9k-1}{k+1}\right)$$

P lies on $y-x+2=0$

$$\frac{9k-1}{k+1} - \left(\frac{8k+3}{k+1}\right) + 2 = 0$$

$$\frac{9k-1-8k-3+2k+2}{k+1} = 0$$

$$3k-2=0$$

$$k = 2/3$$

Ratio $\frac{2}{3} : 1$

$\boxed{2:3}$ Ans.

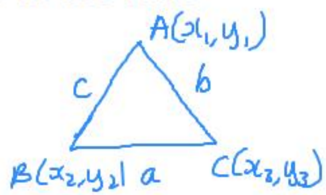
Centres of a Triangle

CENTROID AND INCENTRE :

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the coordinates of the centroid are : $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$ ✓

& the coordinates of the incentre are : ✓ $\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$

Note that incentre divides the angle bisectors in the ratio
$(b+c) : a$; $(c+a) : b$ & $(a+b) : c$.



(G) Centroid — Medians
(I) Incentre — Angle Bisectors
(C) Circumcentre — ⊥ Bisectors
(O) Orthocentre — Altitudes

* O G C Property

- For isosceles triangle centroid, circumcentre, orthocentre and incentre are collinear.
- For a triangle Orthocentre (O), Centroid (G), Circumcentre (C) are collinear and centroid divides orthocentre and circumcentre in the ratio 2 : 1 internally.

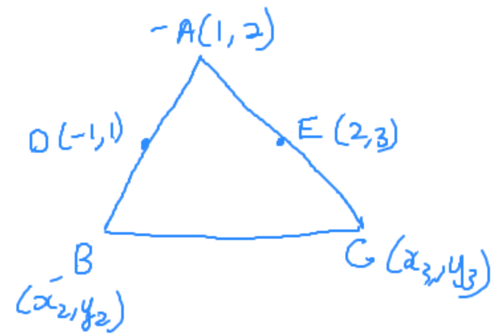
$\underset{2}{O\ G}\ \underset{1}{C}$

- For equilateral Δ, centroid, circumcentre, orthocentre and incentre coincide.

Problems

A triangle has a vertex at (1, 2) and the mid-points of the two sides through it are (–1, 1) and (2, 3). Then, the centroid of this triangle is **(2019 Main, 12 April II)**

(a) $\left(1, \frac{7}{3}\right)$ (b) $\left(\frac{1}{3}, 2\right)$ (c) $\left(\frac{1}{3}, 1\right)$ (d) $\left(\frac{1}{3}, \frac{5}{3}\right)$



$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$

$-1 = \frac{x_2+1}{2} \Rightarrow x_2 = -3$

$1 = \frac{2+y_2}{2} \Rightarrow y_2 = 0$

$2 = \frac{x_3+1}{2} \Rightarrow x_3 = 3$

$3 = \frac{y_3+2}{2} \Rightarrow y_3 = 4$

B(–3, 0) C(3, 4) A(1, 2)

$G\left(\frac{-3+3+1}{3}, \frac{0+4+2}{3}\right)$

$G\left(\frac{1}{3}, 2\right)$

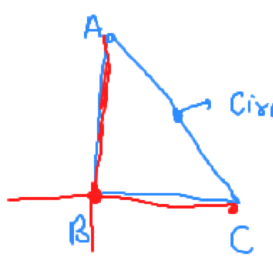
Problems

A B C

The circumcentre of the triangle with vertices (0, 0), (3, 0) and (0, 4) is -

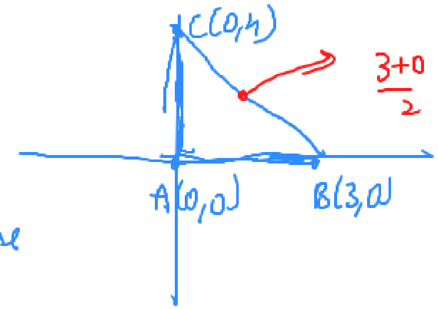
(A) (1, 1) (B) (2, 3/2) (C) (3/2, 2) (D) none of these

Right Angle Triangle



Circumcentre is mid point of Hypotenuse

Orthocentre is at the vertex having 90° angle



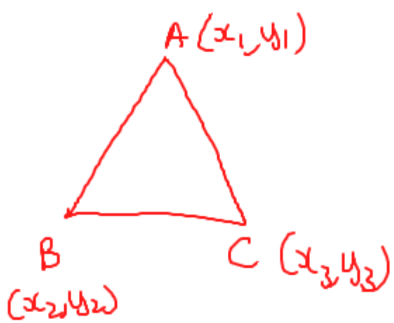
$\frac{3+0}{2}, \frac{0+4}{2} = \frac{3}{2}, \frac{4}{2}$

$\frac{3}{2}, 2$

AREA OF A TRIANGLE :

The area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\,|\,x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\,| = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$



Every Δ

$$\begin{cases} \Delta_1 = \frac{1}{2} \times \text{base} \times \text{height} \\ \Delta_2 = \sqrt{s(s-a)(s-b)(s-c)} \\ \Delta_3 = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B \end{cases}$$

$$\Delta = \frac{\sqrt{3}}{4} a^2$$

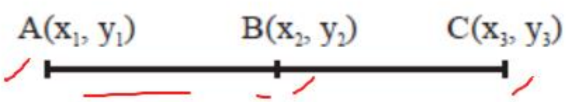
Equilateral Triangle

COLLINEARITY OF THREE POINTS :

Different conditions for three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ to be collinear are as follows

(i) $AB + BC = AC$, $AC - AB = BC$

(ii)



Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

(iii) If the area of triangle ABC be zero then the three points will be collinear.

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Problems

Find the value of x so that the points (x, –1), (2, 1) and (4, 5) are collinear.

Area of $\Delta = 0$

$$\frac{1}{2}\left[x(1-5) + 2(5+1) + 4(-1-1)\right] = 0$$

$$-4x + 12 - 8 = 0$$

$$4x = 4$$

$$x = 1$$

LOCUS :

The curve described by a point which moves under given condition or conditions is called its locus.

EQUATION TO LOCUS OF A POINT :

The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point.

Steps to find locus of a point.

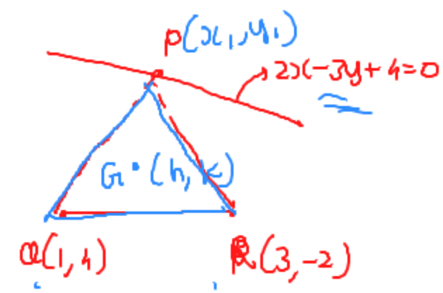
Step I : Assume the coordinates of the point say (h, k) whose locus is to be determined.
Step II: Write the given condition in mathematical form involving h, k.
Step III : Eliminate the variable (s), if any.
Step IV : Replace h by x and k by y in the result obtained in step III.
The equation so obtained is the locus of the point which moves under some condition(s).

Problems

A point P moves on the line $2x - 3y + 4 = 0$. If $Q(1, 4)$ and $R(3, -2)$ are fixed points, then the locus of the centroid of ΔPQR is a line (2019 Main, 10 Jan I)

(a) with slope $\frac{2}{3}$ (b) with slope $\frac{3}{2}$

(c) parallel to Y-axis (d) parallel to X-axis



$h = \frac{x_1 + 1 + 3}{3}$

$\boxed{x_1 = 3h - 4}$

$k = \frac{y_1 + 4 - 2}{3}$

$\boxed{y_1 = 3k - 2}$

P satisfies the given line

$2x_1 - 3y_1 + 4 = 0$

$2(3h-4) - 3(3k-2) + 4 = 0$

$6h - 8 - 9k + 6 + 4 = 0$

$\boxed{6h - 9k + 2 = 0}$

$h \to x$

$k \to y$

$\boxed{6x - 9y + 2 = 0}$ locus.

$9y = 6x + 2 \Rightarrow y = \frac{6x}{9} + \frac{2}{9}$

EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS :

(i) **Slope – intercept form:** $y = mx + c$ is the equation of a straight line whose slope is m & which makes an intercept c on the y–axis.

(ii) **Slope one point form:** $y - y_1 = m(x - x_1)$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1).

m $P(x_1, y_1)$

$$\boxed{y - y_1 = m(x - x_1)}$$

m

m   c

a   b

(iii) **Parametric form :** The equation of the line in parametric form is given by $\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$ (say). Where 'r' is the distance of any point (x , y) on the line from the fixed point (x_1, y_1) on the line. r is positive if the point (x, y) is on the right of (x_1, y_1) and negative if (x, y) lies on the left of (x_1, y_1) .

$\boxed{\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r}$

$P(x_1, y_1)$

$\theta \to$ angle of inclination

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = \pm r$$

where, r is the distance of any point on the line from the given point $A(x_1, y_1)$.

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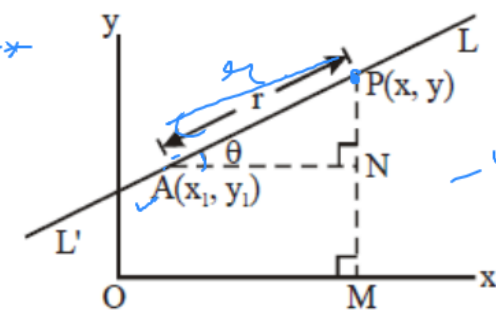


**

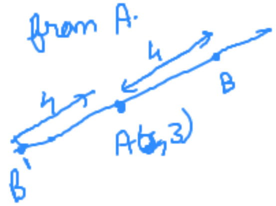
$\therefore x = x_1 + r\cos\theta$

$\therefore y = y_1 + r\sin\theta$

$A(2,3)$ $\theta = \pi/4$

5

4 units from A.



$x = 2 \pm 4\cos 45^\circ$

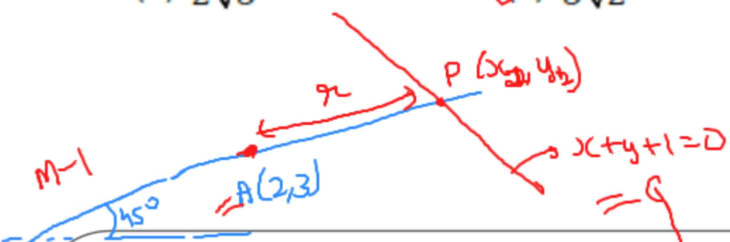
$y = 3 \pm 4\sin 45^\circ$

$\left(x = 2 + \frac{4}{\sqrt{2}},\ y = 3 + \frac{4}{\sqrt{2}}\right)$ & $\left(x = 2 - \frac{4}{\sqrt{2}},\ y = 3 - \frac{4}{\sqrt{2}}\right)$

Problems

Equation of a line which passes through point A(2, 3) and makes an angle of 45 with x axis. If this line meet the line x + y + 1 = 0 at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$



M-2

$y - y_1 = m(x - x_1)$

$y - 3 = 1(x - 2)$

$y - 3 = x - 2$

$x - y + 1 = 0$

$x + y + 1 = 0$

$2x + 2 = 0 \Rightarrow x = -1, \; y = 0$

$P(-1, 0), \; A(2, 3)$

$AP = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

Parametric

$$\left.\begin{array}{l} x_2 = x_1 + r\cos\theta = 2 + \frac{r}{\sqrt{2}} \\ y_2 = y_1 + r\sin\theta = 3 + \frac{r}{\sqrt{2}} \end{array}\right\} P$$

$$2 + \frac{r}{\sqrt{2}} + 3 + \frac{r}{\sqrt{2}} + 1 = 0 \Rightarrow 6 + \frac{2r}{\sqrt{2}} = 0 \Rightarrow r = \frac{-6 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = -3\sqrt{2}$$

Ans: $3\sqrt{2}$

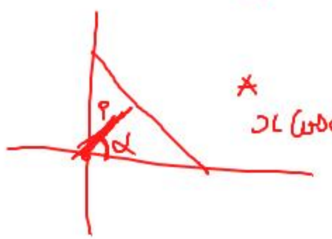
(iv) **Two point form :** $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ is the equation of a straight line which passes through the points (x_1, y_1) & (x_2, y_2).

$A(x_1, y_1) \longrightarrow B(x_2, y_2)$

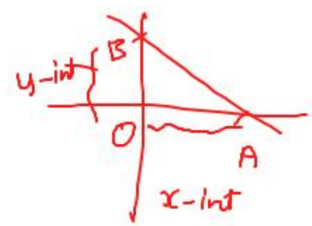
(v) **Intercept form :** $\frac{x}{a} + \frac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively.

(vi) **Perpendicular form :** $x\cos\alpha + y\sin\alpha = p$ is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes angle α with positive side of x–axis.

(vii) **General Form :** $ax + by + c = 0$ is the equation of a straight line in the general form



* $x\cos\alpha + y\sin\alpha = p$



$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$

Problems

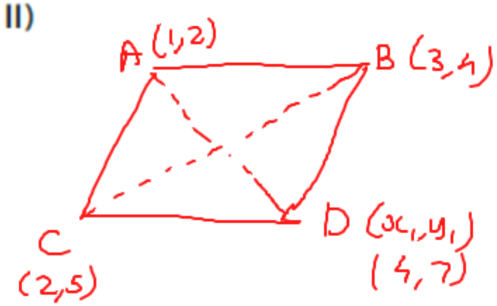
If in a parallelogram $ABDC$, the coordinates of A, B and C are respectively (1, 2), (3, 4) and (2, 5), then the equation of the diagonal AD is **(2019 Main, 11 Jan II)**

(a) $3x + 5y - 13 = 0$

(b) $3x - 5y + 7 = 0$

(c) $5x - 3y + 1 = 0$

(d) $5x + 3y - 11 = 0$



2 Point Form

$y - 2 = \frac{7-2}{4-1}(x-1)$

$y - 2 = \frac{5}{3}(x-1)$

$3y - 6 = 5x - 5 \Rightarrow \boxed{5x - 3y + 1 = 0}$

Diagonals of a ||gram bisect each other

mid point of AD = mid point of BC

$\frac{x_1+1}{2} = \frac{3+2}{2} \Rightarrow x_1 = 4$

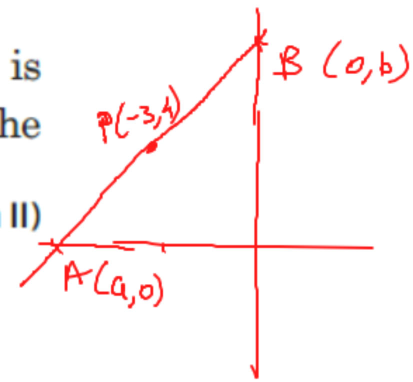
$\frac{y_1+2}{2} = \frac{5+5}{2} \Rightarrow y_1 = 7$

Problems

If a straight line passing through the point $P(-3, 4)$ is such that its intercepted portion between the coordinate axes is bisected at P, then its equation is **(2019 Main, 12 Jan II)**

(a) $x - y + 7 = 0$
(b) $4x - 3y + 24 = 0$ ✓
(c) $3x - 4y + 25 = 0$
(d) $4x + 3y = 0$

P is mid point of AB



$\frac{x}{a} + \frac{y}{b} = 1$

Eqn of line

$\frac{x}{-6} + \frac{y}{8} = 1$

$-4x + 3y = 24$

$4x - 3y + 24 = 0$

$\frac{a+0}{2} = -3 \Rightarrow a = -6$

$\frac{0+b}{2} = 4 \Rightarrow b = 8$