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Problem Solving on Matrices & Determinants

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ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A then the adjoint of A, denoted by adjA, is defined as the transpose of the cofactor matrix.

Then, $\text{adj}A = [C_{ij}]^T \Rightarrow \text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{23} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$C_{ij} = (-1)^{i+j} M_{ij}$

Minor

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 4 \\ 2 & 6 & 7 \end{bmatrix} \text{ find (adj. A)}$$

$\hookrightarrow$ 2x2

If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ then Adj A $= \begin{bmatrix} s & -q \\ -r & p \end{bmatrix}$ e.g. $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ the adj. $A = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

Hence adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing the signs of the off diagonal elements.

$\det(A \cdot B) = \det(A) \cdot \det(B)$

PROPERTIES OF ADJOINT :

Theorem-1 : $A \cdot (\text{adj. } A) = (\text{adj. } A).A = |A|\, I_n$ where A is any square matrix

$|A|\, |\text{adj} A| = |\,|A| I_n|$ $\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}_3$ $|A|^n$

Theorem-2 : Let A be a non-singular matrix of order n. Then

$|\text{adj } A| = |A|^{n-1}$ (**Note:** in particular for n = 3 $|\text{adj. } A| = |A|^2$)

Theorem-3 : If A is a non singular square matrix, then $\boxed{|A| \neq 0}$

(a) $\text{adj}(\text{adj } A) = |A|^{n-2} A$ (b) $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

$|\text{adj}(\text{adj}(A))| = |\,|A|^{n-2} A\,| = |A|^{n \cdot (n-2)} |A|^1$

$= |A|^{n^2 - 2n + 1}$

PROPERTIES OF ADJOINT :

$\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$

$\boxed{AB \neq BA}$

$\text{adj}(A^T) = (\text{adj } A)^T$

$\text{adj}(KA) = K^{n-1} (\text{adj } A)$, K is a scalar

$\text{adj}(2A) = 2^{n-1} \text{adj}(A)$

9) If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to **(2013 Main)**

(a) 4 (b) 11 (c) 5 (d) 0

$$\boxed{|adj(A)| = |A|^{n-1}}$$

$$|adj(A)| = |A|^2$$

$$|adj(A)| = 16$$

$$\cancel{1(0)} - \alpha(4-6) + 3(4-6) = 16$$

$$2\alpha - 6 = 16$$

$$2\alpha = 22$$

$$\boxed{\alpha = 11}$$

INVERSE OF A MATRIX (Square Matrix)

A square matrix A said to be invertible if and only if it is non-singular (i.e. $|A| \neq 0$) and there exists a matrix B such that, $AB = I = BA$.

B is called the **inverse** (reciprocal) of A and is denoted by $\mathbf{A^{-1}}$. Thus

$A^{-1} = B \Leftrightarrow AB = I = BA$

$$\boxed{A^{-1}A = I = AA^{-1}}$$

$$\boxed{A^{-1} = \frac{(\text{adj } A)}{|A|}}$$

~~AB ≠ BA~~

$$A\,.\,(\text{adj } A) = |A|\; I_n$$

$$A^{-1}\,A\;(\text{adj } A) = A^{-1}\,|A|\,I_n$$

$$I_n\,.(\text{adj } A) = A^{-1}\;|A|\;I_n$$

$$A^{-1} = \frac{(\text{adj } A)}{|A|}$$

$\text{adj}(A) = A^{-1}|A|$

$$\boxed{A^{-1} = \frac{\text{adj}(A)}{|A|}}$$

9) If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \dfrac{\pi}{12}$, is equal to

(2019 Main, 9 Jan I)

(a) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

(b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(c) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(d) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

$\frac{50\pi}{12} = 4\pi + \frac{2\pi}{12} = \frac{\pi}{6}$

$\begin{bmatrix} \cos\pi/6 & \sin\pi/6 \\ -\sin\pi/6 & \cos\pi/6 \end{bmatrix}$

$(A^{50})^{-1} = (A^{-1})^{50}$

$A \cdot A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos2\theta & -\sin2\theta \\ \sin2\theta & \cos2\theta \end{bmatrix} = A^2$

$(A^{50}) = \begin{bmatrix} \cos50\theta & -\sin50\theta \\ \sin50\theta & \cos50\theta \end{bmatrix} = \dfrac{\begin{bmatrix} \cos50\theta & \sin50\theta \\ -\sin50\theta & \cos50\theta \end{bmatrix}}{1}$

PROPERTIES OF INVERSE:

Theorem–1 : Every invertible matrix possesses a unique inverse.

Theorem-2 : If A is an invertible square matrix, then A^T is also invertible and
$$(A^T)^{-1} = (A^{-1})^T$$

Theorem-3 : If A is a non-singular matrix, then prove that

$$|A^{-1}| = |A|^{-1} \quad \text{i.e.} \quad \boxed{|A^{-1}| = \frac{1}{|A|}}$$

$\cancel{\left|\frac{1}{A}\right| = \frac{1}{|A|}}$

Theorem-4 : If A & B are invertible matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$.

$$\boxed{|A| \neq 0, \ |B| \neq 0}$$

Property 5 :

$$(A^k)^{-1} = (A^{-1})^k$$

Q)

If $B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$ is the inverse of a 3×3 matrix A, then the sum of all values of α for which $\det(A) + 1 = 0$, is

(2019 Main, 12 April I)

(a) 0 (b) -1 (c) 1 (d) 2

$B = \begin{bmatrix} 5 & 2\alpha & 1 \\ 0 & 2 & 1 \\ \alpha & 3 & -1 \end{bmatrix}$

$A^{-1} = B$

$|A^{-1}| = \frac{1}{|A|}$

$|A| + 1 = 0$

$|A| = -1$

$|A^{-1}| = -1$

$\boxed{|B| = -1}$

$|B| = 5(-5) - 2\alpha(-\alpha) + 1(-2\alpha)$

$|B| = -25 + 2\alpha^2 - 2\alpha$

$0 = 2\alpha^2 - 2\alpha - 24$

$\alpha = \frac{\cancel{2}}{\cancel{2}}$ 1

Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to

(a) M^2 (b) $-N^2$ (c) $-M^2$ (d) MN **(2011)**

$M^T = -M$

$N^T = -N$

$M^2N^2(M^T.N)^{-1}(MN^{-1})^T$

$M^2N^2N^{-1}(M^T)^{-1}(MN^{-1})^T$

$M^2N.(NN^{-1})(-M)^{-1}(MN^{-1})^T$

$M^2N(-M)^{-1}(N^{-1})^T(M^T)$

$M^2N(-M)^{-1}(N^T)^{-1}(-M)$

$M^2N(-M)^{-1}(-N)^{-1}(-M)$

$-M^2NM^{-1}N^{-1}.M$

$-M^2N(NM)^{-1}.M$

$-M^2N.(MN)^{-1}.M$

$-M^2NN^{-1}.M^{-1}.M$

$-M^2$

9)

The total number of matrices $A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}$, $(x, y \in R, x \neq y)$ for which $A^T A = 3I_3$ is

(2019 Main, 9 April II)

(a) 2 (b) 4 (c) 3 (d) 6

$A = \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & 2x & 2x \\ 2y & y & -y \\ 1 & -1 & 1 \end{bmatrix}$

$\begin{bmatrix} 0 & 2x & 2x \\ 2y & y & -y \\ 1 & -1 & +1 \end{bmatrix} \begin{bmatrix} 0 & 2y & 1 \\ 2x & y & -1 \\ 2x & -y & 1 \end{bmatrix} = \begin{bmatrix} 8x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$\begin{bmatrix} 8x^2 & 0 & 0 \\ 0 & 6y^2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$6y^2 = 3$, $8x^2 = 3$

$y = \pm \frac{1}{\sqrt{2}}$, $x = \pm \sqrt{3/8}$

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9) If A is a symmetric matrix and B is a skew-symmetric matrix such that $A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$, then AB is equal to

(2019 Main, 12 April I)

(a) $\begin{bmatrix} -4 & -2 \\ -1 & 4 \end{bmatrix}$

~~(b)~~ $\begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -2 \\ 1 & -4 \end{bmatrix}$

(d) $\begin{bmatrix} -4 & 2 \\ 1 & 4 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -1 & -4 \end{bmatrix}$$

$A^T = A$

$B^T = -B$

$A + B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix}$

$A^T + B^T = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

$A - B = \begin{bmatrix} 2 & 5 \\ 3 & -1 \end{bmatrix}$

$2A = \begin{bmatrix} 4 & 8 \\ 8 & -2 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 3 \\ 5 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 4 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

The set of all values of λ for which the system of linear equations $x-2y-2z=\lambda x, x+2y+z=\lambda y$ and $-x-y=\lambda z$ has a non-trivial solution **(2019 Main, 12 Jan II)**

(a) contains exactly two elements.
(b) contains more than two elements.
(c) is a singleton. (one - element)
(d) is an empty set.

$(0,0,0)$

Trivial — $D \neq 0$, $D=0$ — Infinite

$D_1 = D_2 = D_3 = 0$

$x(1-\lambda)-2y-2z=0$

$x+(2-\lambda)y+z=0$

$-x-y-\lambda z=0$

$$\begin{vmatrix} (1-\lambda) & -2 & -2 \\ 1 & (2-\lambda) & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$\boxed{\lambda = 1}$

$(1-\lambda)^3$

$(1-\lambda)\left[(2-\lambda)(-\lambda)+1\right]+2(-\lambda+1)-2(-1+(2-\lambda))$

$(1-\lambda)\left[\lambda^2-2\lambda+1\right]+\cancel{2(1-\lambda)}-\cancel{2(1-\lambda)}$

$(1-\lambda)(\lambda-1)^2$

$(1-\lambda)(1-\lambda)^2$

If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$, then the inverse of $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is

(2019 Main, 9 April I)

(a) $\begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ 13 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -12 \\ 0 & 1 \end{bmatrix}$

Let A and B be two invertible matrices of order 3×3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det(BA^{-1}B^T)$ is equal to **(2019 Main, 11 Jan II)**

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{16}$ (d) 16