Projectile Motion
&
Circular Motion

JEE and NEET CRASH COURSE

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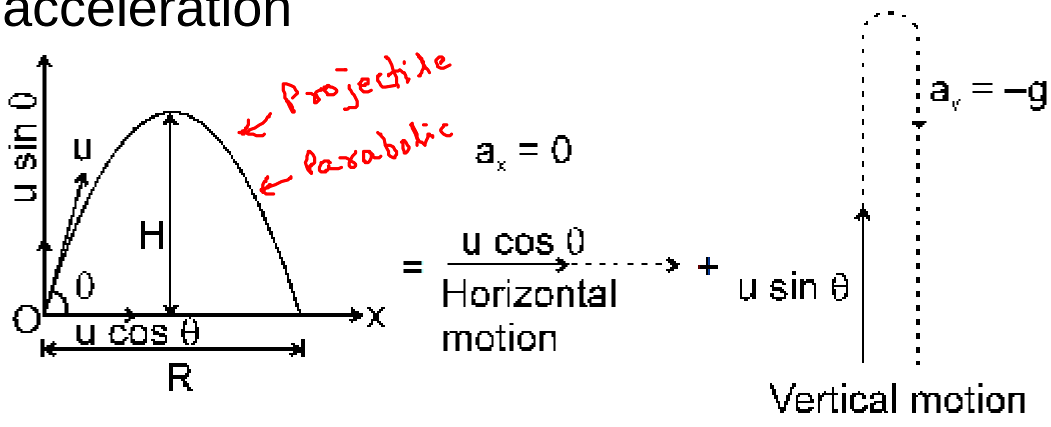
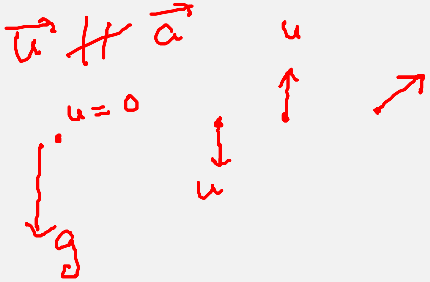
PROJECTILE MOTION

It is two dimensional motion with constant acceleration



		Horizontal direction	Vertical direction
(a)		Initial velocity $u_x = u\cos\theta$	Initial velocity $u_y = u\sin\theta$
(b)		Acceleration $a_x = 0$	Acceleration $a_y = -g$
(c)		Velocity after time t, $v_x = u\cos\theta$	Velocity after time t, $v_y = u\sin\theta - gt$
(d)		Displacement, $x = u\cos\theta t$	Displacement, $y = u\sin\theta t - \frac{1}{2}gt^2$

If $\vec{a}$ = const.
the particle can have only two types of paths.
(i) st. line
if $u = 0$ or $\vec{u} \parallel \vec{a}$ $(\theta = 0^\circ$ or $180^\circ)$
(ii) parabolic
$\vec{u} \nparallel \vec{a}$



Time of Flight, Range and Max. Height

Time of flight :

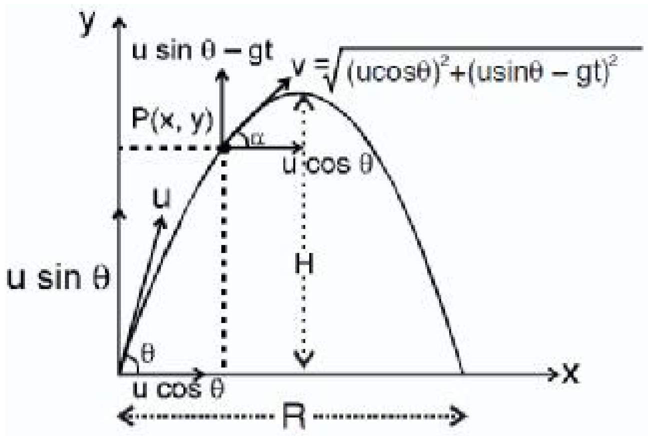
The displacement along vertical direction is zero for the complete flight. Hence, along vertical direction net displacement = 0

$\Rightarrow \quad (u \sin \theta)\, T - \frac{1}{2} g T^2 = 0 \quad \Rightarrow \quad \mathbf{T = \frac{2u\sin\theta}{g}}$

Horizontal range :

$R = u_x . T \quad > \quad R = u \cos \theta . \frac{2u\sin\theta}{g}$

$$\mathbf{R = \frac{u^2 \sin 2\theta}{g}}$$



Maximum height :

At the highest point of its trajectory, particle moves horizontally, and hence vertical component of velocity is zero.

Using 3rd equation of motion i.e. $v^2 = u^2 + 2as$

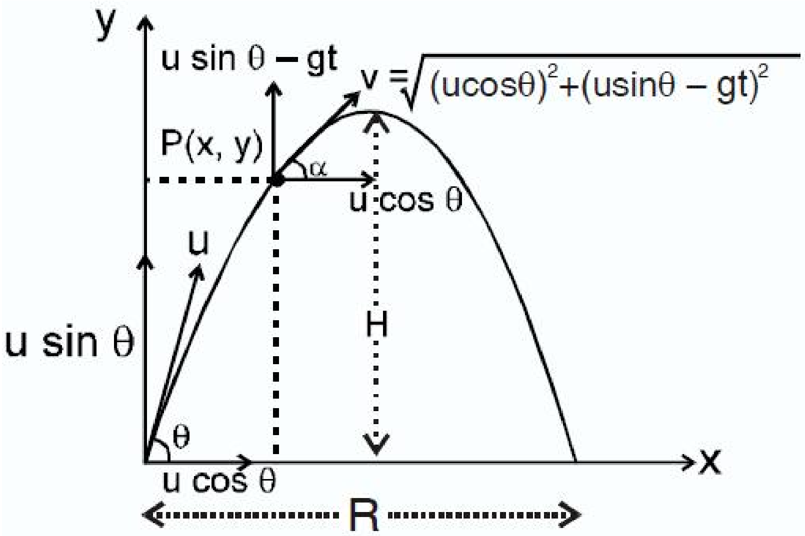
we have for vertical direction $\quad 0 = u^2 \sin^2 \theta - 2gH \quad \Rightarrow \quad \mathbf{H = \frac{u^2 \sin^2 \theta}{2g}}$

Resultant Velocity at time t

$$\vec{v} = v_x\hat{i} + v_y\hat{j} \quad = u\cos\theta\ \hat{i} + (u\sin\theta - gt)\ \hat{j}$$

Where, $|\ \vec{v}\ | = \sqrt{u^2\cos^2\theta + (u\sin\theta - gt)^2}$ and $\tan\alpha = v_y / v_x$

Also, $v\cos\alpha = u\cos\theta \quad \Rightarrow v = \dfrac{u\cos\theta}{\cos\alpha}$



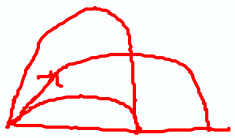
Important Results

- For maximum range $\theta = 45^\circ$

  $$R_{max} = \frac{u^2}{g} \Rightarrow H_{max} = \frac{R_{max}}{2}$$

H is max. $\theta = 90^\circ$

$$H_{max} = \frac{u^2}{2g}$$



- We get the same range for two angle of projections α and $(90 - \alpha)$ but in both cases, maximum heights attained by the particles are different.

  This is because, $R = \frac{u^2 \sin 2\theta}{g}$, and $\sin 2(90 - \alpha) = \sin 180 - 2\alpha = \sin 2\alpha$

- If $R = H$

  i.e. $$\frac{u^2 \sin 2\theta}{g} = \frac{u^2 \sin^2 \theta}{2g}$$

  $\Rightarrow \tan\theta = 4$

- Range can also be expressed as

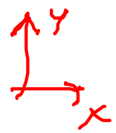
  $$R = \frac{u^2 \sin 2\theta}{g} = \frac{2u \sin\theta . u\cos\theta}{g} = \frac{2u_x u_y}{g}$$

$$T = \frac{2u\sin\theta}{g} = \frac{2u_y}{g}$$

$$H = \frac{u^2 \sin^2\theta}{2g} = \frac{u_y^2}{2g}$$

$$R = \frac{2u_x u_y}{g}$$

Ex1- $\vec{u} = \underset{u_x}{2\hat{i}} + \underset{u_y}{4\hat{j}}$



Find T, R, H

Sol $T = \frac{2u_y}{g} = \frac{2\times4}{10} = 0.8\,s$

$R = \frac{2u_x u_y}{g} = \frac{2\times2\times4}{10} = 1.6\,m$

$H = \frac{u_y^2}{2g} = \frac{16}{20} = 0.8\,m$

Equation of Trajectory

The path followed by a particle (here projectile) during its motion is called its **Trajectory**. Equation of trajectory is the relation between instantaneous coordinates (Here x & y coordinate) of the particle.

If we consider the horizontal direction, $x = u_x . t = u \cos\theta . t$ ...(1)

For vertical direction : $y = u_y . t - 1/2\, gt^2 = u \sin\theta . t - 1/2\, gt^2$ ...(2)

Eliminating 't' from equation (1) & (2)

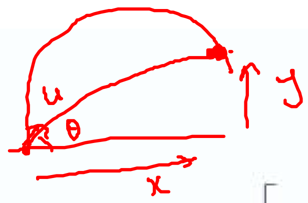
$$y = u \sin\theta . \frac{x}{u\cos\theta} - \frac{1}{2} g \left(\frac{x}{u\cos\theta}\right)^2 \Rightarrow \boxed{y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}}$$

Other forms of trajectory equation :

- $$y = x\tan\theta - \frac{gx^2(1+\tan^2\theta)}{2u^2}$$

- $$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} \Rightarrow y = x\tan\theta\left[1 - \frac{gx^2}{2u^2\cos^2\theta\tan\theta}\right]$$

$$\Rightarrow y = x\tan\theta\left[1 - \frac{gx}{2u^2\sin\theta\cos\theta}\right] \Rightarrow \boxed{y = x\tan\theta\left[1 - \frac{x}{R}\right]}$$



Ex $y = 16x - 4x^2$
Find angle of projection & range

Sol $y = 16x\left[1 - \frac{x}{4}\right]$

$\tan\theta = 16$

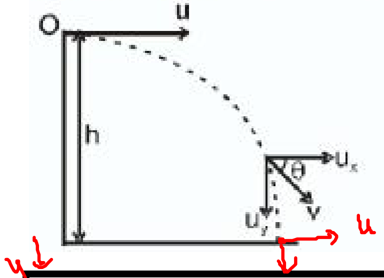
$R = 4$

$x = \frac{R}{2} = 2$

$y_{max} = 32 - 16 = 16\,m$

Projectile thrown horizontally for some Height

Consider a projectile thrown from point O at some height h from the ground with a velocity u. Now we shall study the characteristics of projectile motion by resolving the motion along horizontal and vertical directions.



	Horizontal direction	Vertical direction
(i)	Initial velocity $u_x = u$	Initial velocity $u_y = 0$
(ii)	Acceleration $a_x = 0$	Acceleration $a_y = g$ (downward)

$v_x = u$ $\qquad$ $v_y = u_y + a_y t = g t$

$x = ut$ $\qquad$ $y = \frac{1}{2} g t^2$

Time of flight : $t = \sqrt{\frac{2h}{g}}$

Horizontal range : $R = u\sqrt{\frac{2h}{g}}$

Velocity with which the projectile hits the ground : $V = \sqrt{u^2 + 2gh}$

$v_y^2 = u_y^2 + 2a_y y$

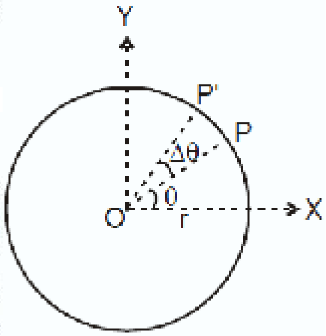
$= 2gH$

$v = \sqrt{v_x^2 + v_y^2}$

Kinematics of Circular Motion

(a) Angular Position :

The angular position of the particle P at a given instant may be described by the angle θ between OP and OX. This angle θ is called the **angular position** of the particle.



(b) Angular Displacement :

Definition: Angle through which the position vector of the moving particle rotates in a given time interval is called its angular displacement. Angular displacement depends on origin, but it does not depends on the reference line. As the particle moves on above circle its angular position θ changes. Suppose the point rotates through an angle $\Delta\theta$ in time Δt, then $\Delta\theta$ is angular displacement.

Important points :

- Angular displacement is a dimensionless quantity. Its SI unit is radian, some other units are degree and revolution

$$2\pi \text{ rad} = 360° = 1 \text{ rev}$$

Angular Velocity

(i) Average Angular Velocity

$$\omega_{av} = \frac{\text{Angular displacement}}{\text{Total time taken}}$$

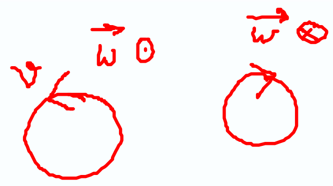
$$\boxed{\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}}$$

where θ_1 and θ_2 are angular position of the particle at time t_1 and t_2. Since angular displacement is a scaler, average angular velocity is also a scaler.

(ii) Instantaneous Angular Velocity

It is the limit of average angular velocity as Δt approaches zero. i.e.

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$$



Since infinitesimally small angular displacement $\overrightarrow{d\theta}$ is a vector quantity, instantaneous angular velocity $\vec{\omega}$ is also a vector, whose direction is given by right hand thumb rule.

Important points :

- Angular velocity has dimension of $[T^{-1}]$ and SI unit rad/s.
- If a body makes 'n' rotations in 't' seconds then average angular velocity in radian per second will be

$$\omega_{av} = \frac{2\pi n}{t}$$

If T is the period and 'f' the frequency of uniform circular motion $\omega_{av} = \frac{2\pi}{T} = 2\pi f$

Angular Acceleration

(i) Average Angular Acceleration :

Let ω_1 and ω_2 be the instantaneous angular speeds at times t_1 and t_2 respectively, then the average angular acceleration α_{av} is defined as

$$\vec{\alpha}_{av} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1} = \frac{\Delta\vec{\omega}}{\Delta t}$$

(ii) Instantaneous Angular Acceleration :

It is the limit of average angular acceleration as Δt approaches zero, i.e., $\vec{\alpha} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt}$

since $\vec{\omega} = \frac{d\vec{\theta}}{dt}$, $\therefore$ $\vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$, Also $\vec{\alpha} = \omega\frac{d\vec{\omega}}{d\theta}$

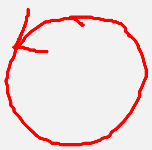
$a = v\frac{dv}{dx} = \frac{d^2x}{dt^2}$

Important points :

- Both average and instantaneous angular acceleration are axial vectors with dimension $[T^{-2}]$ and unit rad/s^2.
- If $\alpha = 0$, circular motion is said to be uniform. $\vec{\omega} =$ const $\left(T = \frac{2\pi R}{v} = \frac{2\pi}{\omega}\ ;\ a_c = \frac{v^2}{R} = \omega^2 R\right)$

$\vec{\alpha} \parallel \vec{\omega}$ if $\vec{\omega}\uparrow$

$\vec{\alpha}$ opp. to $\vec{\omega}$ if $\vec{\omega}\downarrow$



$\vec{\omega}$ ⊙

$\vec{\omega}$ ⊗

Motion with Constant Angular Acceleration

$\omega_0 \Rightarrow$ Initial angular velocity
$\omega \Rightarrow$ Final angular velocity
$\alpha \Rightarrow$ Constant angular acceleration
$\theta \Rightarrow$ Angular displacement

Circular motion with constant angular acceleration is analogous to one dimensional translational motion with constant acceleration. Hence even here equation of motion have same form.

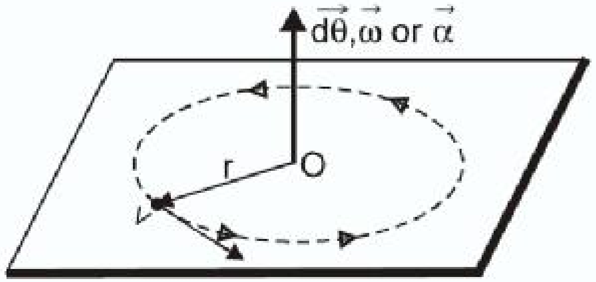
$$\omega = \omega_0 + \alpha t$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta$$

$$\theta = \left(\frac{\omega + \omega_0}{2}\right) t$$

$$\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}\overset{(2n-1)}{(\cancel{\theta_n - \theta_{n-1}})} = \theta_n - \theta_{n-1}$$



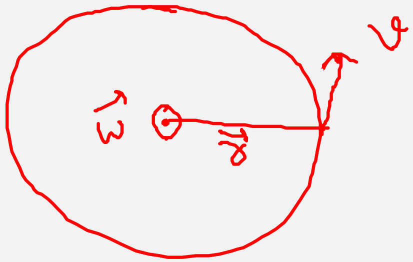
Relation b/w Velocity & Angular Velocity

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

Here, $\vec{v}$ is velocity of the particle, $\vec{\omega}$ is angular velocity about centre of circular motion and '$\vec{r}$' is position of particle w.r.t. center of circular motion.

Since $\vec{\omega} \perp \vec{r}$

$\boxed{v = \omega r \text{ for circular motion.}}$



Radial and Tangential Acceleration

There are two types of acceleration in circular motion ; Tangential acceleration and centripetal acceleration.

(a) Tangential acceleration :-

Component of acceleration directed along tangent of circle is called tangential acceleration. It is responsible for changing the speed of the particle. It is defined as,

$$a_t = \frac{dv}{dt} = \frac{d|\vec{v}|}{dt} = \text{Rate of change of speed.}$$

$$a_t = \alpha r$$

$$\frac{d}{dt}(\omega r) = \frac{d\omega}{dt} r$$

IMPORTANT POINT

(i) In vector form $\vec{a}_t = \vec{\alpha} \times \vec{r}$

(ii) If tangential acceleration is directed in direction of velocity then the speed of the particle increases.

(iii) If tangential acceleration is directed opposite to velocity then the speed of the particle decreases.

(b) Centripetal acceleration :-

It is responsible for change in direction of velocity. In circular motion, there is always a centripetal acceleration.

Centripetal acceleration is always variable because it changes in direction.

$$a_r = \frac{v^2}{R} = \omega^2 R$$

Centripetal acceleration is also called radial acceleration or normal acceleration.

Total Acceleration

(c) Total acceleration :

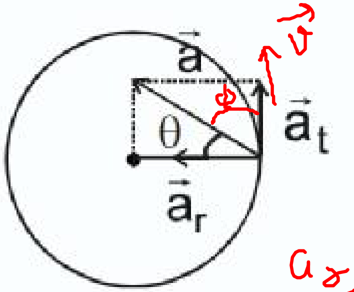
Total acceleration is vector sum of centripetal acceleration and tangential acceleration.

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{a}_r + \vec{a}_t$$

$$a = \sqrt{a_t^2 + a_r^2}$$

$$\tan\theta = \frac{a_r}{a_t}$$

$$\tan\theta = \frac{a_t}{a_r}$$



$$\tan\phi = \frac{a_r}{a_t}$$

IMPORTANT POINT

(i) Differentiation of speed gives tangential acceleration.

(ii) Differentiation of velocity ($\vec{v}$) gives total acceleration.

(iii) $\left|\frac{d\vec{v}}{dt}\right|$ & $\frac{d|\vec{v}|}{dt}$ are not same physical quantity. $\left|\frac{d\vec{v}}{dt}\right|$ is the magnitude of rate of change of velocity, i.e.

Dynamics of Circular Motion

If there is no force acting on a body it will move in a straight line (with constant speed). Hence if a body is moving in a circular path or any curved path, there must be some force acting on the body.

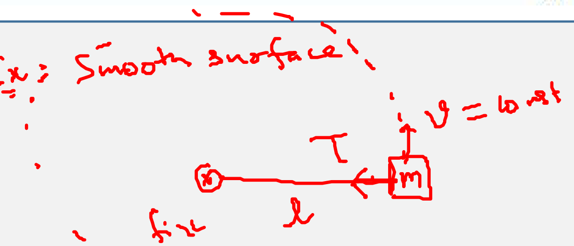
If speed of body is constant, the net force acting on the body is along the inside normal to the path of the body and it is called centripetal force.

$$\text{Centripetal force } (F_c) = ma_c = \frac{mv^2}{r} = m\,\omega^2 r$$

However if speed of the body varies then, in addition to above centripetal force which acts along inside normal, there is also a force acting along the tangent of the path of the body which is called tangential force.

$$\text{Tangential force } (F_t) = Ma_t = M\frac{dv}{dt} = M\,\alpha\,r \quad ; \quad \text{where } \alpha \text{ is the angular acceleration}$$

Ex: Smooth surface



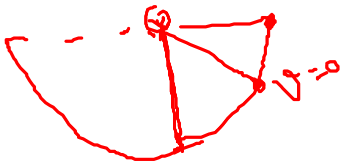
$T = \frac{mv^2}{l}$

Find tension in the string

Motion in Vertical Circle

v_{min}, T_{min}

$0 \le v_b \le \sqrt{2g\ell}$

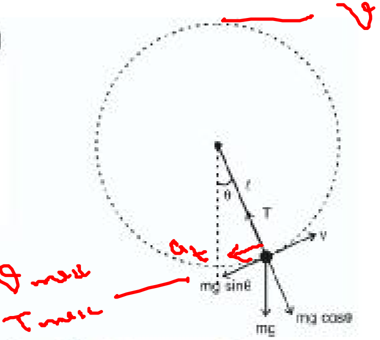


Let us consider the motion of a point mass tied to a string of length ℓ and whirled in a vertical circle.

Applying Newton's law along radial direction

$$T - mg\cos\theta = m.a_r = \frac{mv^2}{\ell}$$

or $$T = \frac{mv^2}{\ell} + mg\cos\theta \quad .....(1)$$



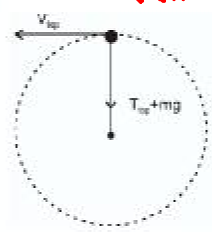
v_{max} T_{max}

$a_t = g\sin\theta$

Hence condition for completing the circle (or looping the loop) is $T_{min} \ge 0$ or $T_{top} \ge 0$.

v_{min} at top for complete circle $\sqrt{g\ell}$

$$T_{top} + mg = \frac{mv_{top}^2}{\ell} \quad ....(2)$$

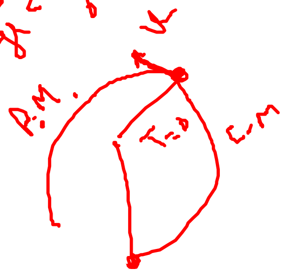


Equation...(2) could also be obtained by putting $\theta = \pi$ in equation ..(1).

For looping the loop, $T_{top} \ge 0$.

$$\rightarrow \quad \frac{mv_{top}^2}{\ell} \ge mg \quad \rightarrow \quad v_{top} \ge \sqrt{g\ell} \quad ....(3)$$

$\sqrt{2g\ell} < v_b < \sqrt{5g\ell}$

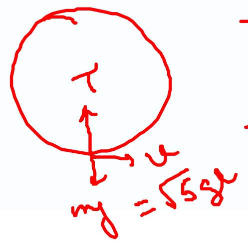


If speed at the lowest point is u, then from conservation of mechanical energy between lowest point and top most point.

$$\frac{1}{2}mu^2 = \frac{1}{2}mv_{top}^2 + mg.2\ell$$

using equation ..(3) for v_{top} we get $\quad u \ge \sqrt{5g\ell}$

i.e., for looping the loop, velocity at lowest point must be $\ge \sqrt{5g\ell}$.



$T - mg = \frac{mv^2}{\ell}$

$T = mg + \frac{m}{\ell}.5g\ell$

$= 6mg$

Circular Turning on Roads

When vehicles go through turnings, they travel along a nearly circular arc. There must be some force which will produce the required centripetal acceleration. If the vehicles travel in a horizontal circular path, this resultant force is also horizontal. The necessary centripetal force is being provided to the vehicles by following three ways.

1. **By friction only**
2. **By banking of roads only.**
3. **By friction and banking of roads both.**

By Friction Only

Suppose a car of mass m is moving at a speed v in a horizontal circular arc of radius r. In this case, the necessary centripetal force to the car will be provided by force of friction f acting towards center

Thus, $\quad f = \dfrac{mv^2}{r}$

Further, limiting value of f is μN

or $\quad f_L = \mu N = \mu mg \quad (N = mg)$

Therefore, for a safe turn without sliding $\dfrac{mv^2}{r} \le f_L$

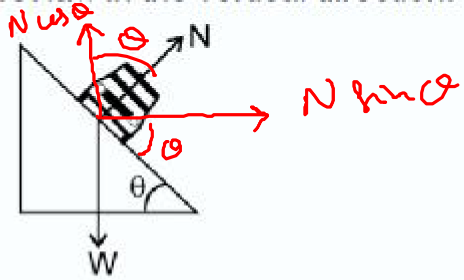
or $\quad \dfrac{mv^2}{r} \le \mu mg \quad$ or $\quad \boxed{\mu \ge \dfrac{v^2}{rg} \text{ or } v \le \sqrt{\mu rg}}$

Here, two situations may arise. If μ and r are known to us, the speed of the vehicle should not exceed $\sqrt{\mu rg}$ and if v and r are known to us, the coefficient of friction should be greater than $\dfrac{v^2}{rg}$.

By Banking of Roads Only

Friction is not always reliable at circular turns if high speeds and sharp turns are involved to avoid dependence on friction, the roads are banked at the turn so that the outer part of the road is some what lifted compared to the inner part.

Applying Newton's second law along the radius and the first law in the vertical direction.



$$N \sin\theta = \frac{mv^2}{r}$$

or $$N \cos\theta = mg$$

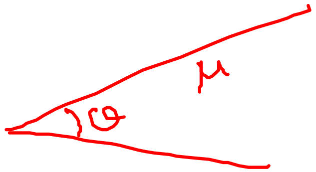
from these two equations, we get

$$\tan\theta = \frac{v^2}{rg} \quad \text{or} \quad v = \sqrt{rg\tan\theta}$$

Note :

- The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle of banking for an aircraft, i.e., the angle through which it should tilt while negotiating a curve, to avoid deviation from the circular path.
- The expression $\tan\theta = \frac{v^2}{rg}$ also gives the angle at which a cyclist should lean inward, when rounding a corner. In this case, θ is the angle which the cyclist must make with the vertical.

By Friction and Banking of Road Both



$\tan\phi = \mu$

ϕ

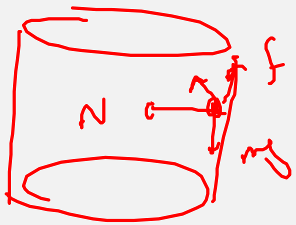
Angle of repose

$$v_{min} = \sqrt{\frac{rg(\tan\theta - \mu)}{(1+\mu\tan\theta)}}$$

$$v_{max} = \sqrt{\frac{rg(\tan\theta + \mu)}{(1-\mu\tan\theta)}}$$

$$v_{min} = \sqrt{rg\tan(\theta-\phi)}$$

$$v_{max} = \sqrt{rg\tan(\theta+\phi)}$$



$$N = \frac{mv^2}{r}$$

$$f_{max} \geq mg$$

$$\mu N \geq mg$$

Additional Concepts

w.r.t person