Matrices

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Definition : Rectangular array of m n numbers enclosed by a pair of brackets

$$\text{e.g. } A = \begin{array}{c} \text{column} \\ \begin{bmatrix} 2 & 3 & 7 \\ 1 & -1 & 5 \end{bmatrix} \end{array} \text{ — Rows}$$

6 elements

order = 2×3

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}$$

a_{ij}

i — row
j = column

m → rows
n → column

$$\boxed{\text{order} = m \times n}$$

Abbreviated as : $A = [a_{ij}]$ $1 \le i \le m$; $1 \le j \le n$, i denotes the row and j denotes the column is called a matrix of order m × n. The elements of a matrix may be real or complex numbers. If all the elements of a matrix are real, the matrix is called real matrix.

order

Note: An m × n matrix has mn elements.

SPECIAL TYPE OF MATRICES :

(a) **Row Matrix** : $A = [a_{11}, a_{12}, ... a_{1n}]$ having one row . $(1 \times n)$ matrix.
(or row vectors)

$1 \times n$

(b) **Column Matrix** : $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ having one column. $(m \times 1)$ matrix
(or column vectors)

$m \times 1$

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times}$

$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

(c) **Zero or Null Matrix :** $(A = O_{m \times n})$
An $m \times n$ matrix all whose entries are zero.

$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 null matrix & $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is 3×3 null matrix

(d) **Horizontal Matrix :** A matrix of order $m \times n$ is a horizontal matrix if $n > m$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 1 & 1 \end{bmatrix}$$

Column > Row

[ ]

~~(e)~~ **Verical Matrix :** A matix of order m × n is a vertical matrix if m > n. $\begin{bmatrix} 2 & 5 \\ 1 & 1 \\ 3 & 6 \\ 2 & 4 \end{bmatrix}$

Note: Every row vertex is also a Horizontal but not the converse.

|||ly every column matrix is also a vertical matrix but not the converse.

$\boxed{\text{Row > column}}$

~~(f)~~ **Square Matrix : (Order n)**

If number of rows = number of columns ⇒ a square matrix. A real square matrix all whose elements are positive is called a positive matrices. Such matrices have application in mechanics and economics.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

$\boxed{\text{order} = 2}$ $\boxed{\text{order} = 3}$

Square Matrix

Triangular Matrix

Upper triangular matrix

if $a_{ij} = 0 \ \forall \ i > j$

$$A = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$$

$a_{ij} = 0 \ \forall \ i > j$

Lower triangular matrix

if $a_{ij} = 0 \ \forall \ i < j$

$$A = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}$$

a_{12} a_{13}

a_{23}

Diagonal Matrix

atleast one, $a_{ii} \neq 0$ and $a_{ij} = 0$ if $i \neq j$

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

Trace of a Matrix $= d_1 + d_2 + d_3$

abbreviated as dia $(d_1, d_2, d_3 \ldots\ldots d_n)$

Scalar matrix

if $d_1 = d_2 = d_3 \ldots\ldots = a \neq 0$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

Unit matrix

(Identity Matrix)

if $d_1 = d_2 = d_3 \ldots\ldots = 1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

EQUALITY OF MATRICES :

The matrices $A = [a_{ij}]$ & $B = [b_{ij}]$ are equal if ,

(i) both have the same order.

$[A]_{2\times 3} \neq [B]_{3\times 4}$

(ii) $a_{ij} = b_{ij}$ for each pair of i & j .

$$\begin{bmatrix} a & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & b \\ c & 3 \end{bmatrix} \qquad \boxed{\begin{matrix} a = 4 \\ b = 1 \\ c = 2 \end{matrix}}$$

ALGEBRA OF MATRICES :

1. **Addition :** $A + B = [a_{ij} + b_{ij}]$ where A & B are of the same type . (same order)

If A and B are square matrices of the same type then, $t_r(A + B) = t_r(A) + t_2(B)$

Addition of matrices is commutative .

i.e. $A + B = B + A$ where A and B must have the same order

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix}$$

MULTIPLICATION OF A MATRIX BY A SCALAR :

If $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$; $kA = \begin{bmatrix} ka & kb & kc \\ kb & kc & ka \\ kc & ka & kb \end{bmatrix}$ i.e. $k(A + B) = kA + kB$

$$2A = \begin{bmatrix} 2a & 2b & 2c \\ 2b & 2c & 2a \\ 2c & 2a & 2b \end{bmatrix}$$

MULTIPLICATION OF MATRICES : (ROW BY COLUMN)

AB exists if, $A = m \times n$ & $B = n \times p$

2×3 3×3

A B is matrix of 2×3

Note that, AB exists, but BA does not $\Rightarrow$ $\boxed{AB \neq BA}$

$B_{3 \times 3}$ $A_{2 \times 3}$

$A_{m \times n} \times B_{n \times p}$ = possible

row column row column

$2 \cdot 3 \neq 3 \cdot 2$

$[AB]_{m \times p}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}_{3 \times 3} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{13} & - & - \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & - & - \end{bmatrix}$$

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DEFINITIONS :

(a) **Idempotent Matrix :** A square matrix is idempotent provide $A^2 = A$. For an idempotent matrix A, $\mathbf{A^n = A}\ \forall\ \mathbf{n \geq 2, n \in N} \Rightarrow A^n = A,\ n \geq 2.$

(b) **Nilpotent Matrix:** A square matrix is said to be nilpotent matrix of index p, $(p \in N)$, if $A^p = \mathbf{O}$, $A^{p-1} \neq \mathbf{O}$ i.e. if p is the least positive integer for which $\mathbf{A^p = O}$, then A is said to be nilpotent of index p.

e.g. **(1)** $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & 3 \end{bmatrix}$ Note that $\mathbf{A}^3 = 0$ but $A^2 \neq 0 \Rightarrow$ index of nilpotency = 3

$A^p = O$

(c) **Involutary Matrix :** If $A^2 = I$, the matrix is said to be an involutary matrix. An involutary matrix is its own inverse.

e.g. (i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

$A^2 = I$

$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

Idempotent

THE TRANSPOSE OF A MATRIX : (CHANGING ROWS & COLUMNS)

Let A be any matrix . Then , $A = [a_{ij}]$ of order $m \times n$

$\Rightarrow A^T$ or $A' = [a_{ji}]$ for $1 \le i \le n$ & $1 \le j \le m$ of order $n \times m$

Thus A^T is obtained by changing its rows into column and columns into row.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 0 \end{bmatrix}_{3\times 2}$$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

Properties of transpose :

If A^T & B^T denote the transpose of A and B ,

(a) $(A+B)^T = A^T+B^T$; note that A & B have the same order.

(b) $(A\ B)^T = B^T\ A^T$ (Reversal law) A & B are conformable for matrix product AB

Note : In general : $(A_1 . A_2 ,\ldots\ldots\ldots\ldots A_n)^T = A_n^T . \ldots\ldots\ldots\ldots . A_2^T . A_1^T$ (reversal law for transpose)

(c) $(A^T)^T = A$

(d) $(kA)^T = kA^T$, k is a scalar.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

SYMMETRIC & SKEW SYMMETRIC MATRIX :

(a) Symmetric matrix :

A square matrix $A = [a_{ij}]$ is said to be, symmetric if, $a_{ij} = a_{ji}$ $\forall$ i & j (conjugate elements are equal).

Hence for symmetric matrix $\boxed{A = A^T.}$ $a_{ij} = a_{ji}$

Note : Max. number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

(b) Skew symmetric matrix

Square matrix $A = [a_{ij}]$ is said to be skew symmetric if $a_{ij} = -a_{ji}$ $\forall$ i & j (the pair of conjugate elements additive inverse of each other). For a skew symmetric matrix $\boxed{A = -A^T.}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A = A^T$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\boxed{\begin{array}{lll} A^T = A & & \text{Symmetric} \\ A^T = -A & & \text{Skew-Symmetric} \end{array}}$$

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \rightarrow \text{Symmetric}$$

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} = A$$

Note :

(i) If A is skew symmetric, then $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0 \ \forall\ i$. Thus the diagonal elements of a skew square matrix are all zero, but not the converse.

(ii) The determinant value of odd order skew symmetric matrix is zero.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

a_{11} a_{22} a_{33}

$-a = a$

$2a = 0$

$a = 0$

$-d = d$

$d = 0$

3

ADJOINT OF A SQUARE MATRIX :

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be cofactor of a_{ij} in A then the adjoint of A, denoted by adjA, is defined as the transpose of the cofactor matrix.

Then, $\text{adj}A = [C_{ij}]^T \Rightarrow \text{adj}A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{23} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$

$\boxed{2\times 2,\ 3\times 3}$

$C = (-1)^{i+j} M_{ij}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 4 \\ 2 & 6 & 7 \end{bmatrix} \text{ find (adj. A).}$$

$M_{ij} =$

$$\begin{matrix} +C_{11} & -C_{12} & +C_{13} \\ -C_{21} & +C_{22} & -C_{23} \\ +C_{31} & -C_{32} & +C_{33} \end{matrix}$$

$$C = \begin{bmatrix} -24 & -27 & 30 \\ 4 & 1 & -2 \\ 8 & 11 & -10 \end{bmatrix}^T$$

$$\text{adj} = C^T = \begin{bmatrix} -24 & 4 & 8 \\ -27 & 1 & 11 \\ 30 & -2 & -10 \end{bmatrix}$$