3D
Geometry

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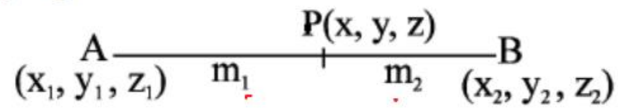
Definition

A General :

(1) Distance (d) between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



(2) **Section Fomula**

$$x = \frac{m_2 x_1 + m_1 x_2}{m_1 + m_2} \quad ; \quad y = \frac{m_2 y_1 + m_1 y_2}{m_1 + m_2} \quad ; \quad z = \frac{m_2 z_1 + m_1 z_2}{m_1 + m_2}$$

(For external division take –ve sign)

DIRECTION COSINES :

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ the angles which this vector makes with the +ve directions OX, OY & OZ are called **DIRECTION ANGLES** & their cosines are called the **DIRECTION COSINES**.

$\cos\alpha = \frac{a_1}{|\vec{a}|} = l$, $\cos\beta = \frac{a_2}{|\vec{a}|}$, $\cos\Gamma = \frac{a_3}{|\vec{a}|}$

Note that, $\cos^2\alpha + \cos^2\beta + \cos^2\Gamma = 1$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

+ve x-axis $\hat{i}$

$\cos\alpha = \frac{\vec{a}\cdot\hat{i}}{|\vec{a}|\,|\hat{i}|} = \frac{a_1}{|\vec{a}|}$

Direction Cosine and direction ratio's of a line

(3) Direction cosine of a line has the same meaning as d.c's of a vector.

(a) Any three numbers a, b, c proportional to the direction cosines are called the direction ratios i.e.

$$\boxed{\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}}$$

same sign either +ve or –ve should be taken through out.

note that d.r's of a line joining x_1, y_1, z_1 and x_2, y_2, z_2 are proportional to $x_2 - x_1$, $y_2 - y_1$ and $z_2 - z_1$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

D.C's $= \frac{a_1}{|\vec{a}|}, \frac{a_2}{|\vec{a}|}, \frac{a_3}{|\vec{a}|}$

D.R's $= a_1, a_2, a_3$

$A(x_1, y_1, z_1)$ $B(x_2, y_2, z_2)$

$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

(b) If θ is the angle between the two lines whose d.c's are l_1, m_1, n_1 and l_2, m_2, n_2

$$\boxed{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2}$$

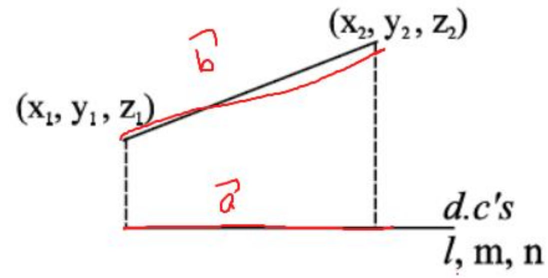
hence if lines are perpendicular then $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

if lines are parallel then $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

note that if three lines are coplanar then $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$

$\vec{a} \wedge \vec{b} = \theta$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \dfrac{l_1l_2+m_1m_2+n_1n_2}{1}$



(4) Projection of the join of two points on a line with d.c's l, m, n are $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

Projection of $\vec{b}$ on $\vec{a}$ is

$\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|} \rightarrow$

$(x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}$ $\quad \vec{a} = l\hat{i} + m\hat{j} + n\hat{k}$

Problems

The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$, is (2014 Main)

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

I II

l_1, m_1, n_1 l_2, m_2, n_2

$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$

$l + m + n = 0 \Rightarrow l = -(m+n)$

$l^2 = m^2 + n^2$

$(-(m+n))^2 = m^2 + n^2$

$\cancel{m^2} + \cancel{n^2} + 2mn = \cancel{m^2} + \cancel{n^2}$

$mn = 0$

Case-1

$m = 0, \; l = -n$

$(-n, 0, n) \equiv (-1, 0, 1)$ D.R.1

Case-2

$n = 0, \; l = -m$

$(-m, m, 0) \equiv (-1, 1, 0)$ DR2

$\cos\theta = \frac{-1\times-1 + 0\times1 + 1\times0}{\sqrt{2}\times\sqrt{2}}$

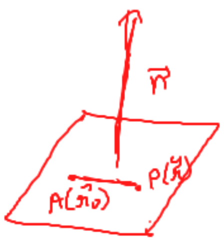
$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

EQUATION OF A PLANE :

The equation $(\vec{r} - \vec{r}_0).\vec{n} = 0$ represents a plane containing the point with p.v. $\vec{r}_0$ where $\vec{n}$ is a vector normal to the plane. $\vec{r}.\vec{n} = d$ is the general equation of a plane.

$\boxed{(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0}$ Eqn of plane.



$\vec{AP} = \vec{r} - \vec{r}_0$

$\vec{n}$ is $\perp$

$\vec{AP} \cdot \vec{n} = 0$

$ax+by+c=0$

B PLANE

(i) General equation of degree one in x, y, z i.e. $ax + by + cz + d = 0$ represents a plane.

(ii) Equation of a plane passing through (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where a, b, c are the direction ratios of the normal to the plane.

(iii) Equation of a plane if its intercepts on the co-ordinate axes are x_1, y_1, z_1 is

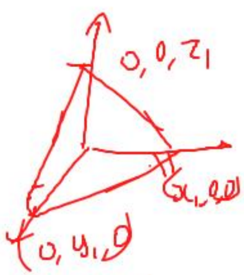
$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1.$$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r_0} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$

$(\vec{r} - \vec{r_0}) \cdot \vec{n} = 0$



$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$

$\boxed{\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1}$

intercept from of plane

$\boxed{a(x-x_1) + b(y-y_1) + c(z-z_1) = 0}$

passes through (x_1, y_1, z_1)

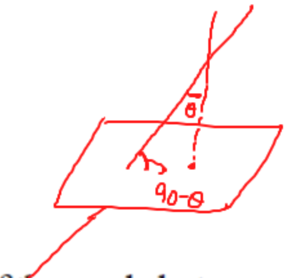
& having d.r. of normal as a, b, c.

(v) **Parallel and perpendicular planes** – Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

a_1, b_1, c_1 & a_2, b_2, c_2

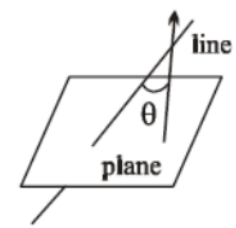
parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$ and

coincident if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = \dfrac{d_1}{d_2}$



(vi) Angle between a plane and a line is the compliment of the angle between the normal to the plane and the line. If $\left.\begin{array}{l}\text{Line} \;\; : \vec{r} = \vec{a} + \lambda \vec{b} \\ \text{Plane} : \vec{r}\,.\,\vec{n} = d\end{array}\right]$ then $\cos(90-\theta) = \sin\theta = \dfrac{\vec{b}\,.\,\vec{n}}{|\vec{b}|\,.\,|\vec{n}|}$.

where θ is the angle between the line and normal to the plane.



Problems

If an angle between the line, $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z-3}{-2}$ and the plane, **(2019 Main, 12 Jan II)**

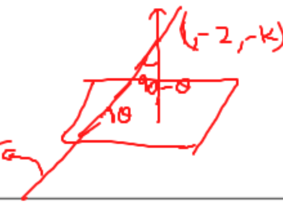
$x - 2y - kz = 3$ is $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$, then value of k is

(a) $\sqrt{\frac{5}{3}}$ (b) $\sqrt{\frac{3}{5}}$ (c) $-\frac{3}{5}$ (d) $-\frac{5}{3}$

$A(-1,2,3)$

$DR \rightarrow 2, 1, -2$

$\sin^{-1}\frac{1}{3}$



$\cos^{-1}\frac{2\sqrt{2}}{3} = \sin^{-1}\frac{1}{3}$

$$\cos(90-\theta) = \frac{\cancel{2\times 1} + \cancel{1\times(-2)} + (-2)(-k)}{3 \times \sqrt{5+k^2}} = \frac{2k}{3\sqrt{5+k^2}} = \sin\theta = \sin\sin^{-1}\frac{1}{3}$$

$$\frac{2k}{\cancel{3}\sqrt{5+k^2}} = \frac{1}{\cancel{3}}$$

$2k = \sqrt{5+k^2}$

$4k^2 = 5 + k^2 \Rightarrow 3k^2 = 5$

$k^2 = 5/3$

$k = \sqrt{\frac{5}{3}}$

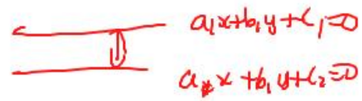
(vii) Length of the perpendicular from a point (x_1, y_1, z_1) to a plane $ax + by + cz + d = 0$ is

$$p = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

(viii) Distance between two parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\left| \frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$p = \left| \frac{ax_1+by_1+c}{\sqrt{a^2+b^2}} \right|$$



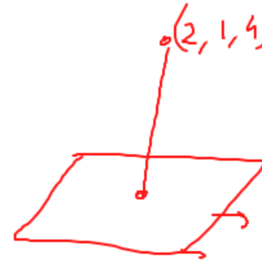
$$\frac{c_1-c_2}{\sqrt{a_1^2+b_1^2}}$$

Problems

$\vec{r} = \vec{a} + \lambda \vec{b}$

The length of the perpendicular drawn from the point (2, 1, 4) to the plane containing the lines $\mathbf{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\mathbf{r} = (\hat{i} + \hat{j}) + \mu(-\hat{i} + \hat{j} - 2\hat{k})$ is (2019 Main, 12 April II)

(a) 3 (b) $\frac{1}{3}$ (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$



$\dfrac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$

plane = ?
$x-y-z=0$

Plane : Point $(1,1,0)$
Normal D.R' $(-1,1,1)$

$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

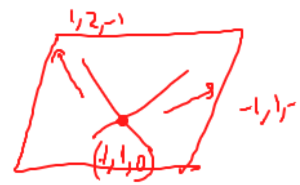
$-1(x-1)+1(y-1)+1(z-0)=0$

$-x+1+y-1+z=0$

$-x+y+z=0$

$\boxed{x-y-z=0}$

$l = \left|\dfrac{2-1-4}{\sqrt{1^2+1^2+1^2}}\right| \quad \dfrac{3}{\sqrt{3}} = \sqrt{3}$

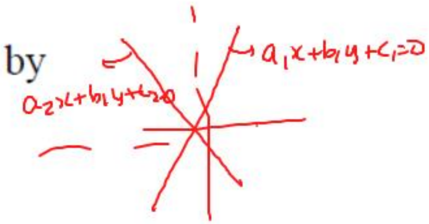


$\vec{n} = \vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -1 & 1 & -2 \end{vmatrix}$

$= \hat{i}(-3) + \hat{j}(3) + \hat{k}(3)$

$= -3\hat{i} + 3\hat{j} + 3\hat{k}$

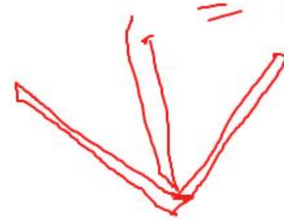
(ix) Planes bisecting the angle between two planes
$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2 + b_2y + c_2z + d_2 = 0$ is given by

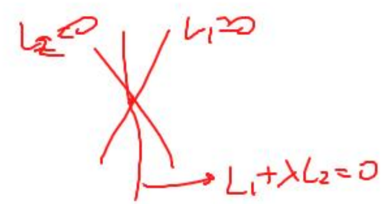


$$\left|\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\right| = \pm\left|\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\right|$$

Of these two bisecting planes , one bisects the acute and the other obtuse angle between the given planes.

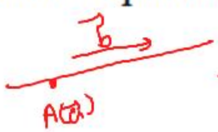
(x) Equation of a plane through the intersection of two planes P_1 and P_2 is given by $P_1 + \lambda P_2 = 0$





VECTOR EQUATION OF A LINE :

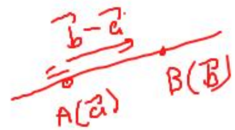
Parametric vector equation of a line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\boxed{\vec{r} = \vec{a} + t(\vec{b} - \vec{a})}$ where t is a parameter. If the line passes through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is, $\boxed{\vec{r} = \vec{a} + t\,\vec{b}}$ **



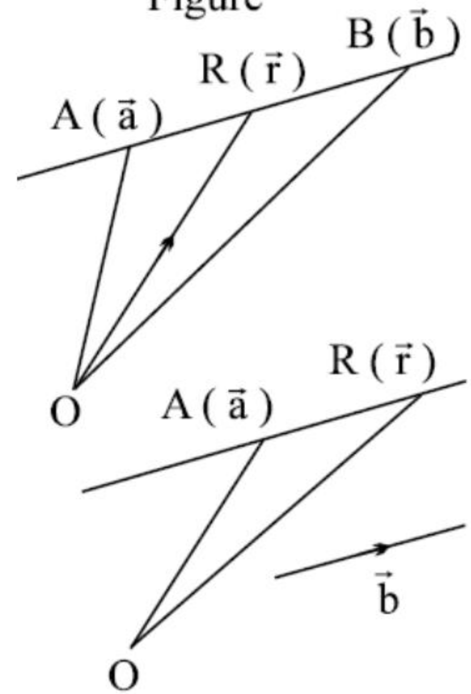
$\vec{r} = \hat{i} + \hat{j} + \lambda(5\hat{i} - 2\hat{j} + \hat{k})$

Note that the equations of the bisectors of the angles between the lines $\vec{r} = \vec{a} + \lambda\,\vec{b}$ & $\vec{r} = \vec{a} + \mu\,\vec{c}$ is :

$\vec{r} = \vec{a} + t\left(\hat{b} + \hat{c}\right)$ & $\vec{r} = \vec{a} + p\left(\hat{c} - \hat{b}\right)$.



Figure



C STRAIGHT LINE IN SPACE

(i) Equation of a line through A (x_1, y_1, z_1) and having direction cosines l, m, n are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

$$\boxed{\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}}$$

2D Line. $ax+by+c=0$

3D Plane $ax+by+cz+d=0$

and the lines through (x_1, y_1, z_1) and (x_2, y_2, z_2)

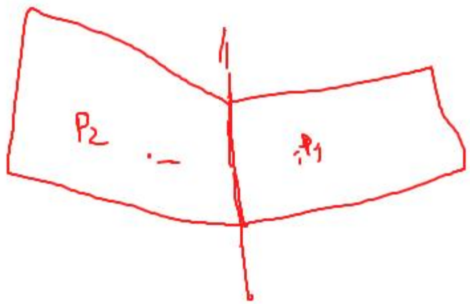
$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

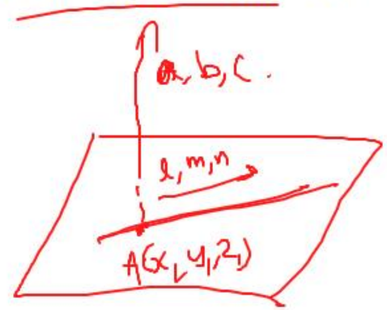
$$\boxed{\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}}$$

(ii) Intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represent the unsymmetrical form of the straight line.

(iii) General equation of the plane containing the line $\dfrac{x - x_1}{l} = \dfrac{y - y_1}{m} = \dfrac{z - z_1}{n}$ is

$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$ where $Al + bm + cn = 0$.



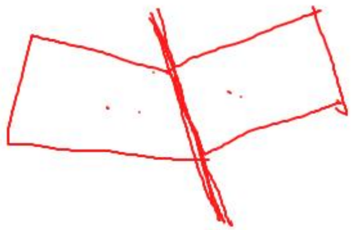


Problems

Let P be the plane, which contains the line of intersection of the planes, $x+y+z-6=0$ and $2x+3y+z+5=0$ and it is perpendicular to the XY-plane. Then, the distance of the point $(0, 0, 256)$ from P is equal to **(2019 Main, 9 April II)**

(a) $63\sqrt{5}$ (b) $205\sqrt{5}$

(c) $\frac{11}{\sqrt{5}}$ (d) $\frac{17}{\sqrt{5}}$



$P_1 + \lambda P_2 = 0$

$(x+y+z-6) + \lambda(2x+3y+z+5) = 0$

$\Rightarrow x(1+2\lambda) + y(1+3\lambda) + z(1+\lambda) + 5\lambda - 6 = 0$

$\hookrightarrow$ $\perp$ to XY plane.

$\vec{n} = (1+2\lambda)\hat{i} + (1+3\lambda)\hat{j} + (1+\lambda)\hat{k}$

$1+\lambda = 0 \Rightarrow \lambda = -1$

$-x - 2y - 11 = 0$

$x + 2y + 11 = 0$ $(0,0,256)$

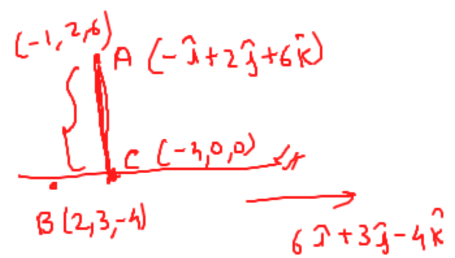
$\frac{0+0+11}{\sqrt{1^2+2^2}} = \frac{11}{\sqrt{5}}$

Problems

The distance of the point having position vector $-\hat{i}+2\hat{j}+6\hat{k}$ from the straight line passing through the point $(2, 3, -4)$ and parallel to the vector, $6\hat{i}+3\hat{j}-4\hat{k}$ is

(2019 Main, 10 April II)

(a) $2\sqrt{13}$ (b) $4\sqrt{3}$ (c) 6 (d) 7



eqn of line = $\vec{r} = 2\hat{i}+3\hat{j}-4\hat{k}+\lambda(6\hat{i}+3\hat{j}-4\hat{k})$

$= (2+6\lambda)\hat{i}+(3+3\lambda)\hat{j}-4(1+\lambda)\hat{k}$

$AC = \sqrt{3^2+2^2+6^2} = 7.$

$C\,(2+6\lambda,\ 3+3\lambda,\ -4-4\lambda)$ $C(-4,0,0)$

D.R of $AC = 3+6\lambda,\ 1+3\lambda,\ -10-4\lambda$

DR of line $6\hat{i}+3\hat{j}-4\hat{k}$

Dot product $=0$

$18+36\lambda+3+9\lambda+40+16\lambda=0$

$61\lambda+61=0$

$\lambda=-1$

Problems

If the line $\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{-1}$ intersects the plane $2x+3y-z+13=0$ at a point P and the plane $3x+y+4z=16$ at a point Q, then PQ is equal to

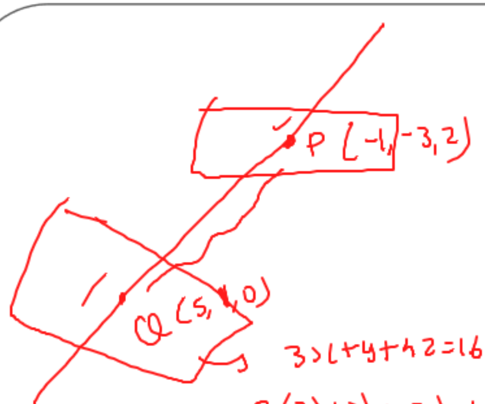
(2019 Main, 12 April I)

(a) 14 (b) $\sqrt{14}$ (c) $2\sqrt{7}$ (d) $2\sqrt{14}$

$(2x+3y-z+13=0)$

$\frac{x-2}{3}=\frac{y+1}{2}=\frac{z-1}{-1}=\lambda$

$x=3\lambda+2,\ y=2\lambda-1,\ z=1-\lambda$



$2(3\lambda+2)+3(2\lambda-1)-(1-\lambda)+13=0$ $x=-1,\ y=-3,\ z=2$

$6\lambda+4+6\lambda-3-1+\lambda+13=0$

$13\lambda+13=0 \Rightarrow \lambda=-1$

$PQ=\sqrt{6^2+4^2+2^2}$

$3x+y+4z=16$

$3(3\lambda+2)+2\lambda-1+4(1-\lambda)=16$ $x=5,\ y=1,\ z=0$ $=\sqrt{56}=2\sqrt{14}$

$9\lambda+6+2\lambda-1+4-4\lambda=16$ $7\lambda=7$

$\lambda=1$