Vector Algebra

Vishal Garg (B.Tech, IIT Bombay)

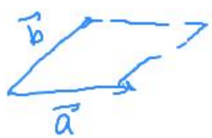
(vi) ☞ Geometrically $|\vec{a}\times\vec{b}|$ = area of the parallelogram whose two adjacent sides are represented by $\vec{a}$ & $\vec{b}$.

(vii) ☞ Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\hat{n} = \pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

☞ A vector of magnitude 'r' & perpendicular to the palne of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

☞ If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

* $\vec{a}\times\vec{b}$ is always ⊥ to plane containing $\vec{a}$ & $\vec{b}$



Area of ||gram = $|\vec{a}\times\vec{b}|$

Problems

Let $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ be two vectors. If a vector perpendicular to both the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ has the magnitude 12, then one such vector is

$$\boxed{\pm 12 \frac{(a+b) \times (a-b)}{|(a+b) \times (a-b)|}}$$

(2019 Main 12 April II)

(a) $4(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

(b) $4(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

(c) $4(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

(d) $4(-2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

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Problems

Let $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + x\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, for some real x. Then $|\mathbf{a} \times \mathbf{b}| = r$ is possible if **(2019 Main, 8 April II)**

(a) $0 < r \le \sqrt{\frac{3}{2}}$ (b) $\sqrt{\frac{3}{2}} < r \le 3\sqrt{\frac{3}{2}}$

(c) $3\sqrt{\frac{3}{2}} < r < 5\sqrt{\frac{3}{2}}$ (d) $r \ge 5\sqrt{\frac{3}{2}}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & x \\ 1 & -1 & 1 \end{vmatrix}$$

$$\vec{a}\times\vec{b} = (x+2)\hat{i} + (x-3)\hat{j} - 5\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(x+2)^2 + (x-3)^2 + 5^2}$$

$$= \sqrt{2x^2 - 2x + 38} = r.$$

$$\sqrt{2x^2-2x+38} \geq \sqrt{\frac{75}{2}}$$

$$r \geq \sqrt{\frac{75}{2}} \Rightarrow \boxed{r \geq 5\sqrt{\frac{3}{2}}}$$

$2x^2 - 2x + 38$

$D = 4 - 4 \times 38 \times 2$

$= 4(1 - 76)$

$= 4 \times (-75)$

$= -300$

$$\frac{-D}{4a} = \frac{300}{8} = \frac{75}{2}$$

$$\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$$

$$2x^2 - 2x + 38 \geq \frac{75}{2}$$

$$\boxed{\sqrt{2x^2-2x+38} \geq \sqrt{\frac{75}{2}}}$$

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SCALAR TRIPLE PRODUCT / BOX PRODUCT / MIXED PRODUCT :

- The scalar triple product of three vectors $\vec{a}, \vec{b}$ & $\vec{c}$ is defined as :

  $\boxed{\vec{a} \times \vec{b} . \vec{c}} = |\vec{a}||\vec{b}||\vec{c}| \sin\theta \cos\phi$ where θ is the angle between $\vec{a}$ & $\vec{b}$ & ϕ is the angle between $\vec{a} \times \vec{b}$ & $\vec{c}$.

  It is also defined as $[\vec{a}\ \vec{b}\ \vec{c}]$, spelled as box product .

- Scalar triple product geometrically represents the volume of the parallelopiped whose three couterminous edges are represented by $\vec{a}, \vec{b}$ & $\vec{c}$ i.e. $V = [\vec{a}\ \vec{b}\ \vec{c}]$

- In a scalar triple product the position of dot & cross can be interchanged i.e.

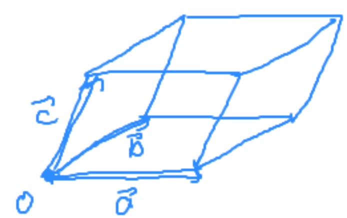
  $\vec{a} . (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) . \vec{c}$ OR $\boxed{[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]}$

  Dot Product
  Cross Product

- $\vec{a} . (\vec{b} \times \vec{c}) = -\vec{a} . (\vec{c} \times \vec{b})$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}]$

$$\boxed{(\vec{a} \times \vec{b}) \cdot \vec{c} = [\vec{a} . \vec{b} . \vec{c}]}$$

$\vec{a} \cdot (\vec{b} \times \vec{c})$



Vol of parallelopiped

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ & $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then $[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

In general , if $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$ & $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \left[\vec{l}\,\vec{m}\,\vec{n}\right]$; where $\vec{\ell}$, $\vec{m}$ & $\vec{n}$ are non coplanar vectors.

- If $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\;\vec{b}\;\vec{c}] = 0$.

$\vec{a} = \vec{l} + \vec{m} + \vec{n}$

$\vec{b} = 2\vec{l} - 3\vec{m} - \vec{n}$

$\vec{c} = -\vec{l} + \vec{n}$

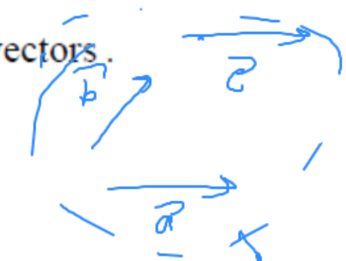
$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ -1 & 0 & 1 \end{vmatrix} [\vec{l}\,\vec{m}\,\vec{n}]$$

$\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} - 3\hat{j} - \hat{k}$

$\vec{c} = -\hat{i} + \hat{k}$

$$[\vec{a}\;\vec{b}\;\vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ -1 & 0 & 1 \end{vmatrix} =$$



Vol = 0

$[\vec{a}\;\vec{b}\;\vec{c}] = 0$

Problems

Let $\alpha \in R$ and the three vectors

$\mathbf{a} = \alpha\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \alpha\hat{\mathbf{k}}$

and $\mathbf{c} = \alpha\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. Then, the set

$S = \{\alpha : \mathbf{a}, \mathbf{b} \text{ and } \mathbf{c} \text{ are coplanar}\}$ **(2019 Main, 12 April II)**

(a) is singleton

(b) is empty

(c) contains exactly two positive numbers

(d) contains exactly two numbers only one of which is positive

$\vec{a}, \vec{b}, \vec{c}$ are coplanar

$\Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\Rightarrow \begin{vmatrix} \alpha & 1 & 3 \\ 2 & 1 & -\alpha \\ \alpha & -2 & 3 \end{vmatrix}$$

$$\alpha(3-2\alpha) + 1(-\alpha^2-6) + 3(-4-\alpha) = 0$$

$$\cancel{3\alpha} - 2\alpha^2 - \alpha^2 - 6 - 12 - \cancel{3\alpha} = 0$$

$$-3\alpha^2 = 18$$

$$\boxed{\alpha^2 = -6}$$ No real soln

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Problems

The sum of the distinct real values of μ, for which the vectors, $\mu\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+\mu\hat{\mathbf{j}}+\hat{\mathbf{k}}$, $\hat{\mathbf{i}}+\hat{\mathbf{j}}+\mu\hat{\mathbf{k}}$ are coplanar, is **(2019 Main, 12 Jan I)**

(a) 2 (b) 0 (c) 1 (d) -1

$$\begin{vmatrix} \mu & 1 & 1 \\ 1 & \mu & 1 \\ 1 & 1 & \mu \end{vmatrix} = 0$$

$$\mu(\mu^2-1)+1(1-\mu)+1(1-\mu)=0$$

$$\mu^3-\mu+1-\mu+1-\mu=0$$

$$\boxed{\mu^3-3\mu+2=0} = (\mu-1)(\mu^2+\mu-2) = (\mu-1)(\mu+2)(\mu-1) = (\mu-1)^2(\mu+2)$$

$$(\mu-1)^2(\mu+2)=0$$

$$\mu = 1, 1, -2$$

$$1+(-2)=-1$$

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Problems

Let $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + (\lambda^2 - 1)\hat{\mathbf{k}}$ be coplanar vectors. Then, the non-zero vector $\mathbf{a} \times \mathbf{c}$ is **(2019 Main, 11 Jan I)**

(a) $-10\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$ (b) $-10\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$

(c) $-14\hat{\mathbf{i}} - 5\hat{\mathbf{j}}$ (d) $-14\hat{\mathbf{i}} + 5\hat{\mathbf{j}}$

$\vec{a} = \hat{i} + 2\hat{j} + 4\hat{k}$

$\vec{c} = 2\hat{i} + 4\hat{j} + 3k$ $(\lambda = 2)$

$[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & \lambda & 4 \\ 2 & 4 & \lambda^2-1 \end{vmatrix} = 0$$

$\boxed{\vec{a} \times \vec{c} = -10\hat{i} + 5\hat{j}}$

$\lambda = 2, 3$ or -3

$\lambda = \pm 3$

$\vec{a} = \hat{i} + 2\hat{j} + 4\hat{k}$

$\vec{c} = 2\hat{i} + 4\hat{j} + 8\hat{k}$

$\vec{a} \times \vec{c} = 0$

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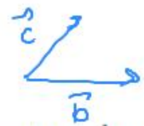
VECTOR TRIPLE PRODUCT :

Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the expression $\boxed{\vec{a} \times (\vec{b} \times \vec{c})}$ is a vector & is called a vector triple product.

$\perp$ to $\vec{b}$ & $\vec{c}$



GEOMETRICAL INTERPRETATION OF $\vec{a} \times (\vec{b} \times \vec{c})$

Consider the expression $\vec{a} \times (\vec{b} \times \vec{c})$ which itself is a vector, since it is a cross product of two vectors $\vec{a}$ & $(\vec{b} \times \vec{c})$. Now $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector perpendicular to the plane containing $\vec{a}$ & $(\vec{b} \times \vec{c})$ but $\vec{b} \times \vec{c}$ is a vector perpendicular to the plane $\vec{b}$ & $\vec{c}$, therefore $\vec{a} \times (\vec{b} \times \vec{c})$ is a vector lies in the plane of $\vec{b}$ & $\vec{c}$ and perpendicular to $\vec{a}$. Hence we can express $\vec{a} \times (\vec{b} \times \vec{c})$ in terms of $\vec{b}$ & $\vec{c}$

i.e. $\vec{a} \times (\vec{b} \times \vec{c}) = x\vec{b} + y\vec{c}$ where x & y are scalars.

- ☞ $\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}}$
- ☞ $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$
- ☞ $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}}$

$\boxed{(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a}}$

Problems

$|\vec{a}| = \sqrt{2^2+1^2+2^2} = 3$

Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c}$ be a vector such that $|\mathbf{c} - \mathbf{a}| = 3$, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$ and the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ is 30°. Then, $\mathbf{a} \cdot \mathbf{c}$ is equal to

(2017 Main)

(a) $\frac{25}{8}$ (b) 2 (c) 5 (d) $\frac{1}{8}$

$|(\vec{a}\times\vec{b})\times\vec{c}| = 3$

$\vec{a}\times\vec{b} = \vec{d} \Rightarrow |\vec{d}\times\vec{c}| = 3$

$|\vec{d}| = |\vec{a}\times\vec{b}|$

$\Rightarrow |\vec{d}||\vec{c}|\sin 30^\circ = 3$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix}$

$\Rightarrow 3 \times |\vec{c}| \times \frac{1}{2} = 3 \Rightarrow |\vec{c}| = 2$

$= 2\hat{i} - 2\hat{j} + \hat{k} \Rightarrow |\vec{a}\times\vec{b}| = \sqrt{2^2+2^2+1^2} = 3$

$\Rightarrow |c-a| = 3$

$|c-a|^2 = 9$

$(c-a)\cdot(c-a) = 9$

$|\vec{c}|^2 + |\vec{a}|^2 - 2\vec{a}\cdot\vec{c} = 9$

$4 + 9 - 2\vec{a}\cdot\vec{c} = 9$

$2\vec{a}\cdot\vec{c} = 4$

$\vec{a}\cdot\vec{c} = 2$

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Problems

Let $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c}$ be a vector such that $\mathbf{a} \times \mathbf{c} + \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{c} = 4$, then $|\mathbf{c}|^2$ is equal to

(2019 Main, 9 Jan I)

(a) 8 ~~(b)~~ $\frac{19}{2}$ (c) 9 (d) $\frac{17}{2}$

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$

$\hat{i}(-1) + \hat{j}(-1) + \hat{k}(2)$

$-\hat{i} - \hat{j} + 2\hat{k}$

$\vec{a} \cdot \vec{a} = |\vec{a}|^2 = (\sqrt{1^2+1^2})^2 = 2$

$\underbrace{\vec{a} \times \vec{c} + \vec{b}} = 0$ $\quad \vec{a} \cdot \vec{c} = 4$ $\quad |\vec{c}|^2$ $\quad \vec{a} \cdot \vec{c} = 4.$

Take cross product with $\vec{a}$

$\vec{a} \times (\vec{a} \times \vec{c}) + \vec{a} \times \vec{b} = 0$

$(\vec{a} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{a})\vec{c} + \vec{a} \times \vec{b} = 0$

$4(\hat{i} - \hat{j}) - 2\vec{c} + (-\hat{i} - \hat{j} + 2\hat{k}) = 0 \Rightarrow 2\vec{c} = 4\hat{i} - 4\hat{j} - \hat{i} - \hat{j} + 2\hat{k}$

$2\vec{c} = 3\hat{i} - 5\hat{j} + 2\hat{k} \Rightarrow \boxed{\vec{c} = \frac{3}{2}\hat{i} - \frac{5}{2}\hat{j} + \hat{k}}$

$|\vec{c}|^2 = \left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + 1^2 = \frac{9}{4} + \frac{25}{4} + 1$

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Problems

Let **a**, **b** and **c** be three unit vectors, out of which vectors **b** and **c** are non-parallel. If α and β are the angles which vector **a** makes with vectors **b** and **c** respectively and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{1}{2}\mathbf{b}$, then $|\alpha - \beta|$ is equal to **(2019 Main, 12 Jan II)**

(a) 30° (b) 45° (c) 90° (d) 60°

Given: $|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$\alpha = \vec{a} \wedge \vec{b}$

$\beta = \vec{a} \wedge \vec{c}$

$\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$

$(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = \frac{1}{2}\vec{b}$

$\vec{a} \cdot \vec{c} = \frac{1}{2}$, $\vec{a} \cdot \vec{b} = 0$

$|\vec{a}||\vec{c}| \cos\beta = \frac{1}{2}$, $|\vec{a}||\vec{b}| \cos\alpha = 0$

$\cos\beta = \frac{1}{2}$, $\cos\alpha = 0$

$\beta = \frac{\pi}{3}$, $\alpha = \frac{\pi}{2}$

Angle b/w two vectors always lie b/w 0 to π.

$|\alpha - \beta| = \left|\frac{\pi}{2} - \frac{\pi}{3}\right| = \frac{\pi}{6} = 30^\circ$

RECIPROCAL SYSTEM OF VECTORS :

If $\vec{a},\vec{b},\vec{c}$ & $\vec{a}',\vec{b}',\vec{c}'$ are two sets of non coplanar vectors such that $\vec{a}.\vec{a}'=\vec{b}.\vec{b}'=\vec{c}.\vec{c}'=1$ then the two systems are called Reciprocal System of vectors.

$3\times\frac{1}{3}=1$

Note : $a'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$; $b'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$; $c'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

$(\vec{a},\vec{b},\vec{c})$ & a', b', c' are said to be reciprocal system if

$a\cdot a'=1$, $b\cdot b'=1$, $c\cdot c'=1$

$a'=\dfrac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $b'=\dfrac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $c'=\dfrac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$