Vector Algebra

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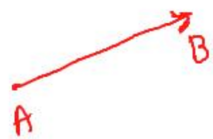
Definition

A VECTOR may be described as a quantity having both magnitude & direction. A vector is generally represented by a directed line segment, say $\vec{AB}$. A is called the **initial point** & B is called the **terminal point**. The magnitude of vector $\vec{AB}$ is expressed by $|\vec{AB}|$.



$|\vec{AB}|$

Types of Vectors

ZERO VECTOR a vector of zero magnitude i.e.which has the same initial & terminal point, is called a **ZERO VECTOR**. It is denoted by O.

UNIT VECTOR a vector of unit magnitude in direction of a vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$.

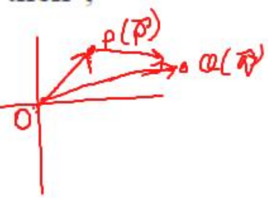
$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

$\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$ $\quad |\vec{a}| = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14}$

$\hat{a} = \frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k})$

POSITION VECTOR let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. If $\vec{a}$ & $\vec{b}$ & position vectors of two point A and B, then ,

$\overrightarrow{AB} = \vec{b} - \vec{a}$ = pv of B − pv of A .



$\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$\overrightarrow{PQ}$ = p.v of Q − p.v of P

$\overrightarrow{PQ}$ = p.v of Q − p.v of P

P = i + j + k

Q = 2i − j

Types of Vectors

Collinear Vectors two vectors are said to be collinear if their directed line segments are parallel disregards to their direction. Collinear vectors are also called **Parallel Vectors**. If they have the same direction they are named as like vectors otherwise unlike vectors.

Simbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{a} = K(\vec{b})$

$a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = K(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$a_1 = Kb_1,\ a_2 = Kb_2,\ a_3 = Kb_3$

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Coplanar Vectors a given number of vectors are called coplanar if their line segments are all parallel to the same plane. Note that **"Two Vectors Are Always Coplanar"**.

Condition of $\vec{a}, \vec{b}, \vec{c}$ to be coplanar.
Scalar Triple product is zero → $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

Problems

Let $\alpha = (\lambda - 2)\,\mathbf{a} + \mathbf{b}$ and $\beta = (4\lambda - 2)\,\mathbf{a} + 3\mathbf{b}$ be two given vectors where vectors **a** and **b** are non-collinear. The value of λ for which vectors α and β are collinear, is (2019 Main, 10 Jan II)

(a) 4 (b) –3 (c) 3 (d) –4

If α & β are collinear. $\vec{\alpha} = k\vec{\beta}$

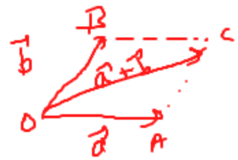
$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$\frac{\lambda - 2}{4\lambda - 2} = \frac{1}{3}$

$3\lambda - 6 = 4\lambda - 2$

$\lambda = -4$.

VECTOR ADDITION :



If two vectors $\vec{a}$ & $\vec{b}$ are represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where OC is the diagonal of the parallelogram OACB.

- $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ (commutative)
- $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ (associativity)
- $\vec{a}+\vec{0}=\vec{a}=\vec{0}+\vec{a}$
- $\vec{a}+(-\vec{a})=\vec{0}=(-\vec{a})+\vec{a}$

MULTIPLICATION OF VECTOR BY SCALARS :

If $\vec{a}$ is a vector & m is a scalar, then m$\vec{a}$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called **Scalar Multiplication**. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then:

$m(\vec{a})=(\vec{a})m=m\vec{a}$

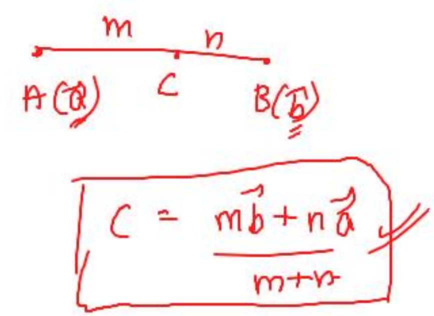
$m(n\vec{a})=n(m\vec{a})=(mn)\vec{a}$

$(m+n)\vec{a}=m\vec{a}+n\vec{a}$

$m(\vec{a}+\vec{b})=m\vec{a}+m\vec{b}$

SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point which divides AB in the ratio m : n is given by : $\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m + n}$. Note p.v. of mid point of AB $= \dfrac{\vec{a} + \vec{b}}{2}$.



SCALAR PRODUCT OF TWO VECTORS :

- $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \, (0 \le \theta \le \pi)$,

note that if θ is acute then $\vec{a}.\vec{b} > 0$ & if θ is obtuse then $\vec{a}.\vec{b} < 0$

- $\vec{a}.\vec{a} = |\vec{a}|^2 = \vec{a}^2$, $\vec{a}.\vec{b} = \vec{b}.\vec{a}$ (commutative)
- $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ (distributive)
- $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}$ $\quad (\vec{a} \ne 0 \quad \vec{b} \ne 0)$
- $\hat{i}.\hat{i} = \hat{j}.\hat{j} = \hat{k}.\hat{k} = 1$; $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$
- projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$.

Note: That vector component of $\vec{a}$ along $\vec{b} = \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$ and perpendicular to $\vec{b} = \vec{a} - \left(\dfrac{\vec{a}\cdot\vec{b}}{\vec{b}^2}\right)\vec{b}$.

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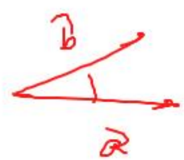
☞ the angle ϕ between $\vec{a}$ & $\vec{b}$ is given by $\cos\phi = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$ $\quad 0 \le \phi \le \pi$

☞ if $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

$|\vec{a}| = \sqrt{a_1^{\,2} + a_2^{\,2} + a_3^{\,2}}$, $|\vec{b}| = \sqrt{b_1^{\,2} + b_2^{\,2} + b_3^{\,2}}$



$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Problems

If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$, are two unit vectors such that $\vec{\mathbf{a}} + 2\vec{\mathbf{b}}$ and $5\vec{\mathbf{a}} - 4\vec{\mathbf{b}}$, are perpendicular to each other, then angle between $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is

$|\vec{a}| = |\vec{b}| = 1$

(a) 45°
(b) 60°
(c) $\cos^{-1}\left(\frac{1}{3}\right)$
(d) $\cos^{-1}\left(\frac{2}{7}\right)$

$(\vec{a} + 2\vec{b}) \cdot (5\vec{a} - 4\vec{b}) = 0$

$5\vec{a}\cdot\vec{a} - 4\vec{a}\cdot\vec{b} + 10\vec{b}\cdot\vec{a} - 8\vec{b}\cdot\vec{b} = 0$

$5|\vec{a}|^2 + 6\vec{a}\cdot\vec{b} - 8|\vec{b}|^2 = 0$

$5 + 6|\vec{a}||\vec{b}|\cos\theta - 8 = 0$

$6\cos\theta = 3$

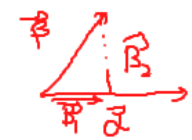
$\cos\theta = 1/2 \quad \theta = 60°$

Problems

Let $\vec{\alpha} = 3\hat{i} + \hat{j}$ and $\vec{\beta} = 2\hat{i} - \hat{j} + 3\hat{k}$. If $\vec{\beta} = \vec{\beta}_1 - \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$, then $\vec{\beta}_1 \times \vec{\beta}_2$ is equal to **(2019 Main, 9 April I)**

(a) $\frac{1}{2}(3\hat{i} - 9\hat{j} + 5\hat{k})$ (b) $\frac{1}{2}(-3\hat{i} + 9\hat{j} + 5\hat{k})$

(c) $-3\hat{i} + 9\hat{j} + 5\hat{k}$ (d) $3\hat{i} - 9\hat{j} - 5\hat{k})$



B_1 is parallel to α

$B_1 = K(3\hat{i} + \hat{j})$

$\beta = B_1 - B_2$

$B_2 = B_1 - \beta$

$B_2 = B_1 - \beta$

$= K(3\hat{i} + \hat{j}) - (2\hat{i} - \hat{j} + 3\hat{k})$

$\vec{B}_2 = (3K-2)\hat{i} + (K+1)\hat{j} - 3\hat{k}$ $\vec{\alpha} = 3\hat{i} + \hat{j}$

$\vec{B}_2 \perp \vec{\alpha}$

$\vec{B}_2 \cdot \vec{\alpha} = 0$

$3(3K-2) + K+1 = 0$

$9K - 6 + K + 1 = 0$

$10K - 5 = 0 \Rightarrow K = \frac{1}{2}$

$\vec{B}_1 = \frac{3}{2}\hat{i} + \frac{3}{2}\hat{j}$

$\vec{B}_2 = -\frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

$$\vec{B}_1 \times \vec{B}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{3}{2} & \frac{3}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & -3 \end{vmatrix}$$

Note :

(i) Maximum value of $\vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|$

(ii) Minimum values of $\vec{a}\,.\,\vec{b} = \vec{a}\,.\,\vec{b} = -|\vec{a}|\,|\vec{b}|$

(iii) Any vector $\vec{a}$ can be written as , $\vec{a} = (\vec{a}\,.\,\hat{i})\,\hat{i} + (\vec{a}\,.\,\hat{j})\,\hat{j} + (\vec{a}\,.\,\hat{k})\,\hat{k}$.

$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

max of $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|$

min of $\vec{a}\cdot\vec{b} = -|\vec{a}||\vec{b}|$

$\vec{a} = (\vec{a}\cdot\hat{i})\,\hat{i} + (\vec{a}\cdot\hat{j})\,\hat{j} + (\vec{a}\cdot\hat{k})\,\hat{k}$

Problems

Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$, $\mathbf{b} = b_1\,\hat{\mathbf{i}} + b_2\,\hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ and $\mathbf{c} = 5\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \sqrt{2}\,\hat{\mathbf{k}}$ be three vectors such that the projection vector of $\mathbf{b}$ on $\mathbf{a}$ is $|\mathbf{a}|$. If $\mathbf{a} + \mathbf{b}$ is perpendicular to $\mathbf{c}$, then $|\mathbf{b}|$ is equal to

(2019 Main, 9 Jan II)

(a) 6 (b) 4 (c) $\sqrt{22}$ (d) $\sqrt{32}$

projection is scalar

projection of $\vec{b}$ on $\hat{a}$ = $\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|} = |\vec{a}|$

$\vec{b}\cdot\vec{a} = |\vec{a}|^2$.

$b_1 + b_2 + 2 = (\sqrt{1+1+2})^2$

$b_1 + b_2 + 2 = 4 \Rightarrow b_1 + b_2 = 2$ — (2)

$(\vec{a}+\vec{b}) \perp \vec{c}$

$\Rightarrow (\vec{a}+\vec{b})\cdot\vec{c} = 0$

$\vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} = 0$

$8 + \vec{b}\cdot\vec{c} = 0$

$8 + 5b_1 + b_2 + 2 = 0$

$5b_1 + b_2 = -10$ — (1)

$b_1 + b_2 = 2$ — (2)

$4b_1 = -12$

$b_1 = -3$

$b_2 = 5$

$\vec{b} = -3\hat{i} + 5\hat{j} + \sqrt{2}k$

$|\vec{b}| = \sqrt{9+25+2} = \sqrt{36} = 6$

VECTOR PRODUCT OF TWO VECTORS :

(i) ☞ If $\vec{a}$ & $\vec{b}$ are two vectors & θ is the angle between them then $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\vec{n}$,

where $\vec{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$ such that $\vec{a}$, $\vec{b}$ & $\vec{n}$ forms a right handed screw system .

(ii) ☞ Lagranges Identity : for any two vectors $\vec{a}$ & $\vec{b}$; $(\vec{a}\times\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

(iii) ☞ Formulation of vector product in terms of scalar product:

$\Rightarrow (\vec{a}\times\vec{b})^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$

The vector product $\vec{a}$ x $\vec{b}$ is the vector $\vec{c}$, such that

(i) $|\vec{c}| = \sqrt{\vec{a}^2\vec{b}^2 - (\vec{a}\cdot\vec{b})^2}$ (ii) $\vec{c}\cdot\vec{a} = 0$; $\vec{c}\cdot\vec{b} = 0$ and

(iii) $\vec{a}$, $\vec{b}$, $\vec{c}$ form a right handed system

(iv) ☞ $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}\ \&\ \vec{b}$ are parallel (collinear) $(\vec{a}\neq 0, \vec{b}\neq 0)$ i.e. $\vec{a}=K\vec{b}$, where K is a scalar.

☞ $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$ (not commutative)

☞ $(m\vec{a})\times\vec{b}=\vec{a}\times(m\vec{b})=m(\vec{a}\times\vec{b})$ where m is a scalar .

☞ $\vec{a}\times(\vec{b}+\vec{c})=(\vec{a}\times\vec{b})+(\vec{a}\times\vec{c})$ (distributive)

☞ $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$ ☞ $\hat{i}\times\hat{j}=\hat{k},\ \hat{j}\times\hat{k}=\hat{i},\ \hat{k}\times\hat{i}=\hat{j}$

(v) ☞ If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$

Problems

Let $\mathbf{a} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ be two vectors. If a vector perpendicular to both the vectors $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ has the magnitude 12, then one such vector is

(2019 Main 12 April II)

(a) $4(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ (b) $4(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$

(c) $4(2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})$ (d) $4(-2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

$4(2\hat{i} - 2\hat{j} - \hat{k})$

$4(-2\hat{i} + 2\hat{j} + \hat{k})$

$\vec{a}$ & $\vec{b}$

$\vec{a} \times \vec{b}$

$\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$

$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$ will be ⊥ to both $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$

$\vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$0 + \vec{b} \times \vec{a} + \vec{b} \times \vec{a} - 0 = 2\vec{b} \times \vec{a}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -2 \\ 3 & 2 & 2 \end{vmatrix}$$

$\hat{i}(8) + \hat{j}(-8) + \hat{k}(-4)$

$\vec{c} = 8\hat{i} - 8\hat{j} - 4\hat{k}$ → one vector ⊥ to $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$

$\hat{c} = \frac{4(2\hat{i} - 2\hat{j} - \hat{k})}{4 \times 3} = \left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right)$

$\pm 12\hat{c} = \pm 12\left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right)$

$= \pm 4(2\hat{i} - 2\hat{j} - \hat{k})$

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