Inverse Trigonometric Functions

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Definition

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ etc. denote angles or real numbers whose sine is x, whose cosine is x and whose tangent is x, provided that the answers given are numerically smallest available. These are also written as arc sinx, arc cosx etc.

$\sin^{-1}x = \theta \Rightarrow \sin\theta = x$

$\cos^{-1}\frac{1}{2} = \theta \Rightarrow \cos\theta = \frac{1}{2}$ $\quad \theta \to$ unique.

Problems

If $0 < \cos^{-1}x < 1$ *and* $1 + \cos^{-1}x + (\cos^{-1}x)^2 + \ldots\ldots\ \infty = 2$ *then* x *is equal to*

(A) $\frac{\pi}{4}$ (B) $\cos\frac{1}{2}$ (C) $\cos\frac{1}{\sqrt{2}}$ (D) $\frac{\pi}{6}$

Infinite G.P $= \frac{a}{1-r}$

$\frac{1}{1-\cos^{-1}x} = 2 \Rightarrow 1-\cos^{-1}x = \frac{1}{2} \Rightarrow \cos^{-1}x = \frac{1}{2}$

$\boxed{x = \cos\frac{1}{2}}$

PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :

(i) $y = \sin^{-1} x$ where $-1 \le x \le 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ and $\sin y = x$.

(ii) $y = \cos^{-1} x$ where $-1 \le x \le 1$; $0 \le y \le \pi$ and $\cos y = x$.

(iii) $y = \tan^{-1} x$ where $x \in R$; $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and $\tan y = x$.

(iv) $y = \text{cosec}^{-1} x$ where $x \le -1$ or $x \ge 1$; $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, $y \ne 0$ and $\text{cosec}\, y = x$.

(v) $y = \sec^{-1} x$ where $x \le -1$ or $x \ge 1$; $0 \le y \le \pi$; $y \ne \frac{\pi}{2}$ and $\sec y = x$.

(vi) $y = \cot^{-1} x$ where $x \in R$, $0 < y < \pi$ and $\cot y = x$.

NOTE THAT : **(a)** 1st quadrant is common to all the inverse functions .

(b) 3rd quadrant is **not used** in inverse functions .

(c) 4th quadrant is used in the **CLOCKWISE DIRECTION** i.e. $-\frac{\pi}{2} \le y \le 0$

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PRINCIPAL VALUES AND DOMAINS OF INVERSE CIRCULAR FUNCTIONS :

$\sin^{-1}\frac{1}{2} = \theta \Rightarrow \sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}$

$\boxed{\sin^{-1}\frac{1}{2} = \frac{\pi}{6}}$

S.No.	Function	Domain	Range
(i)	$y=\sin^{-1}x$	$\underbrace{-1 \le x \le 1}$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
(ii)	$y=\cos^{-1}x$	$-1 \le x \le 1$	$\boxed{0 \le y \le \pi}$
(iii)	$y=\tan^{-1}x$	$x \in R$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
(iv)	$y=\cot^{-1}x$	$x \in R$	$0 < y < \pi$
(v)	$y=\text{cosec}^{-1}x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2},\ y \ne 0$
(vi)	$y=\sec^{-1}x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi\ ;\ y \ne \frac{\pi}{2}$

$\cos^{-1}\frac{1}{2} = \theta \Rightarrow \cos\theta = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{3}}$

$\sin^{-1}3 =$ N.P

NOTE THAT : **(a)** 1st quadrant is common to all the inverse functions .

(b) 3rd quadrant is **not used** in inverse functions .

(c) 4th quadrant is used in the CLOCKWISE DIRECTION i.e. $-\frac{\pi}{2} \le y \le 0$

Problems

The value of $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is equal to

(A) $\frac{\pi}{4}$ (B) $\frac{5\pi}{12}$ (C) $\frac{3\pi}{4}$ (D) $\frac{13\pi}{12}$

$\tan^{-1} 1 = \frac{\pi}{4}$

$[0, \pi]$ $\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$

$\cos^{-1}(-x) = \pi - \cos^{-1} x$

$\sin^{-1}(-x) = -\sin^{-1} x$

Problems

Find the value of

(a) $\sin\left(2\sin^{-1}\frac{3}{5}\right)$

(b) $\cos(2\tan^{-1}2) + \sin(2\tan^{-1}3)$

$\sin\left(2\sin^{-1}\frac{3}{5}\right) = \sin 2\theta = 2\sin\theta\cdot\cos\theta = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$

$\sin^{-1}\frac{3}{5} = \theta \Rightarrow \sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

b) $\tan^{-1}2 = A$, $\tan^{-1}3 = B$

$\tan A = 2$, $\tan B = 3$

$\cos 2A + \sin 2B =$

$\frac{1-\tan^2 A}{1+\tan^2 A} + \frac{2\tan B}{1+\tan^2 B}$

$=$

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Problems

If $\underset{\pi}{\underline{\cos^{-1}x}} + \underset{\pi}{\underline{\cos^{-1}y}} + \underset{\pi}{\underline{\cos^{-1}z}} = 3\pi$, then the value of $x^{2012} + y^{2012} + z^{2012} + \dfrac{6}{x^{2011} + y^{2011} + z^{2011}}$ is equal to

$1 + 1 + 1 + \dfrac{6}{-1-1-1}$

(A) 0   ✓(B) 1   (C) –1   (D) 2

$3 + \dfrac{6}{-3} = 3 - 2 = 1$

$0 \le \cos^{-1}x \le \pi$

Max value of $\cos^{-1}x$ is π.

$\cos^{-1}x = \pi \Rightarrow x = \cos\pi = -1$

$\cos^{-1}y = \pi \Rightarrow y = \cos\pi = -1$

$\cos^{-1}z = \pi \Rightarrow z = \cos\pi = -1$

Properties

P–1 **(i)** $\sin(\sin^{-1}x) = x\ ,\ -1 \le x \le 1$ **(ii)** $\cos(\cos^{-1}x) = x\ ,\ -1 \le x \le 1$

(iii) $\tan(\tan^{-1}x) = x\ ,\ x \in R$ **(iv)** $\sin^{-1}(\sin x) = x\ ,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(v) $\cos^{-1}(\cos x) = x\ ;\ \boxed{0 \le x \le \pi}$ **(vi)** $\tan^{-1}(\tan x) = x\ ;\ -\frac{\pi}{2} < x < \frac{\pi}{2}$

$\sin\theta = \sin(\pi-\theta)$

$f \circ f^{-1}(x) = x$

$f^{-1} \circ f(x) = x.$

$\sin(\sin^{-1}\frac{1}{3}) = 1/3$

$\tan(\tan^{-1}5) = 5$

$\sin^{-1}\sin\frac{\pi}{6} = \frac{\pi}{6}$

$\sin^{-1}\sin\frac{5\pi}{6} = \sin^{-1}\sin(\pi-\frac{\pi}{6}) = \sin^{-1}\sin\frac{\pi}{6} = \frac{\pi}{6}$

$= \sin^{-1}\sin(\frac{5\pi}{6}-\pi+\pi)$

$= \sin^{-1}\sin(-\frac{\pi}{6}+\pi)$

$= \sin^{-1}\sin(\pi-\frac{\pi}{6}) = \sin^{-1}\sin\frac{\pi}{6} = \frac{\pi}{6}$

Is $0 < 3 < \pi$
3.14

$\cos^{-1}\cos\frac{5\pi}{6} = \frac{5\pi}{6}$

$\cos^{-1}\cos 3 = 3$

Problems

$\cos\left(\cos^{-1}\cos\left(\frac{8\pi}{7}\right)+\tan^{-1}\tan\left(\frac{8\pi}{7}\right)\right)$ has the value equal to -

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\frac{6\pi}{7}$ $\quad$ $\pi/7$

(A) 1 $\qquad$ (B) –1 $\qquad$ (C) $\cos\frac{\pi}{7}$ $\qquad$ (D) 0

$[0,\pi]$ $\quad \cos^{-1}\cos\frac{8\pi}{7} = \cos^{-1}\cos\left(\pi+\frac{\pi}{7}\right) = \cos^{-1}\left(-\cos\frac{\pi}{7}\right) = \pi - \cos^{-1}\cos\frac{\pi}{7} = \pi - \frac{\pi}{7} = \frac{6\pi}{7}$

$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ $\quad \tan^{-1}\tan\frac{8\pi}{7} = \tan^{-1}\tan\left(\pi+\frac{\pi}{7}\right) = \tan^{-1}\tan\frac{\pi}{7} = \frac{\pi}{7}$

$\cos\left(\frac{6\pi}{7}+\frac{\pi}{7}\right) = \cos\pi = -1$

Problems

If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to (2019 Main, 9 Jan II)

(a) 0 (b) 10 (c) 7π (d) π

$3\pi = 3 \times 3.14 = 9.42$

$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1}\sin(10) = \sin^{-1}\sin[10 - 3\pi + 3\pi]$

$= \sin^{-1}\sin[3\pi + 10 - 3\pi]$

$= \sin^{-1}[-\sin(10 - 3\pi)]$

$= -[\sin^{-1}\sin(10 - 3\pi)]$

$= -(10 - 3\pi)$

$x = 3\pi - 10$

$[0, \pi]$ $\cos^{-1}(\cos 10) = \cos^{-1}(\cos(10 - 3\pi + 3\pi))$

$= \cos^{-1}\cos(3\pi + 10 - 3\pi)$

$= \cos^{-1} - \cos(10 - 3\pi)$

$= \pi - \cos^{-1}\cos(10 - 3\pi)$

$= \pi - (10 - 3\pi)$

$y = 4\pi - 10$

$y - x = 4\pi - \cancel{10} - 3\pi + \cancel{10}$

$= \pi$

Properties

P–2 **(i)** $\operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}$; $x \le -1 , x \ge 1$

(ii) $\sec^{-1} x = \cos^{-1} \frac{1}{x}$; $x \le -1 , x \ge 1$

(iii) $\cot^{-1} x = \tan^{-1} \frac{1}{x}$; $x > 0$

$= \pi + \tan^{-1} \frac{1}{x}$; $x < 0$

$\operatorname{Cosec}^{-1} 2 = \sin^{-1} \frac{1}{2}$

$\operatorname{Sec}^{-1} (3) = \operatorname{Cos}^{-1} \frac{1}{3}$

$\operatorname{Cot}^{-1} 3 = \tan^{-1} \frac{1}{3}$

$\operatorname{Cot}^{-1} (-3) = \pi + \tan^{-1} \left(\frac{1}{-3}\right)$

Properties

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P–3 ~~(i)~~ $\sin^{-1}(-x) = -\sin^{-1}x$, $-1 \le x \le 1$

~~(ii)~~ $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $-1 \le x \le 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in R$

~~v)~~ $\text{Cosec}^{-1}(-x) = -\text{Cosec}^{-1}x.$

vi) $\text{Sec}^{-1}(-x) = \pi - \text{Sec}^{-1}x.$

$\text{Cos}^{-1}\left(-\frac{1}{2}\right) = \pi - \text{Cos}^{-1}\frac{1}{2}$

P–4 **(i)** $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ $-1 \le x \le 1$

(ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$ $x \in R$

(iii) $\text{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}$ $|x| \ge 1$

Problems

Find the range of $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$.

Domain: $[-1, 1]$

$[-1, 1]$ $\cap$ R

$f(x) = \frac{\pi}{2} + \tan^{-1}x$

$-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$

$\frac{\pi}{2} - \frac{\pi}{2} < \frac{\pi}{2} + \tan^{-1}x < \frac{\pi}{2} + \frac{\pi}{2}$

$\cancel{\boxed{0 < f(x) < \pi}}$

If $-1 \le x \le 1$

$\tan^{-1}(-1) \le \tan^{-1}x \le \tan^{-1}(1)$

$-\frac{\pi}{4} \le \tan^{-1}x \le \frac{\pi}{4}$

$\frac{\pi}{2} - \frac{\pi}{4} \le \frac{\pi}{2} + \tan^{-1}x \le \frac{\pi}{2} + \frac{\pi}{4}$

$\boxed{\frac{\pi}{4} \le f(x) \le \frac{3\pi}{4}}$

Properties

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P–5 $\underbrace{\tan^{-1}x+\tan^{-1}y}=\boxed{\tan^{-1}\frac{x+y}{1-xy}}$ where $x>0\ ,\ y>0\ \&\ xy<1$

$=\pi+\tan^{-1}\frac{x+y}{1-xy}$ where $x>0\ ,\ y>0\ \&\ xy>1$

$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}$ where $x>0\ ,\ y>0$

$\tan^{-1}2+\tan^{-1}3=\ \pi+\tan^{-1}\left(\frac{2+3}{1-2\times3}\right)$

$\tan^{-1}2-\tan^{-1}3\ =\tan^{-1}\left(\frac{2-3}{1+2\times3}\right)$

Problems

If $\tan^{-1}2 + \tan^{-1}4 = \cot^{-1}(\lambda)$ then find λ.

$\tan^{-1}2 + \tan^{-1}4 = \pi + \tan^{-1}\left(\frac{2+4}{1-2\times 4}\right)$

$\cot^{-1}(-x) = \underline{\pi - \cot^{-1}(x)}$

$= \pi + \tan^{-1}\frac{6}{-7}$

$= \pi - \tan^{-1}\frac{6}{7}$.

$= \pi - \cot^{-1}\frac{7}{6}$

$= \cot^{-1}\left(-\frac{7}{6}\right) = \cot^{-1}(\lambda)$

$\lambda = -7/6$.

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Properties

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PROPERTY-6 :

(I) $$\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) \text{if } x\ge 0; y\ge 0 \text{ and } x^2+y^2\le 1\\ \pi-\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right) \text{if } x\ge 0; y\ge 0 \text{ and } x^2+y^2>1\end{cases}$$

note that $x^2+y^2\le 1 \Rightarrow 0\le \sin^{-1}x+\sin^{-1}y\le \frac{\pi}{2}$

(II) ||| ly we have $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right), x>0; y>0$

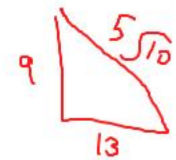
and $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left(xy\mp\sqrt{1-x^2}\sqrt{1-y^2}\right), x>0, y>0, x<y$

Problems

If $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$, $\beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to **(2019 Main, 8 April I)**

(a) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (b) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

(c) $\tan^{-1}\left(\frac{9}{14}\right)$ (d) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$



$\alpha = \cos^{-1}\frac{3}{5}$

$\cos\alpha = \frac{3}{5}$

$\tan\alpha = \frac{4}{3}$

$\alpha = \tan^{-1}\frac{4}{3}$

$\alpha - \beta = \tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{3}$

$= \tan^{-1}\dfrac{\frac{4}{3} - \frac{1}{3}}{1 + \frac{4}{3} \times \frac{1}{3}} = \tan^{-1}\dfrac{1}{13/9} = \tan^{-1}\frac{9}{13} = \sin^{-1}\frac{9}{5\sqrt{10}} = \cos^{-1}\frac{13}{5\sqrt{10}}$

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SUMMATION OF SERIES :

The sum $\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{4n}{n^4 - 2n^2 + 2}\right)$ is equal to

(A) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{2}{3}$ (B) $4\tan^{-1} 1$ (C) $\frac{\pi}{2}$ (D) $\sec^{-1}\left(-\sqrt{2}\right)$

$\boxed{\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}}$

$x>0, y>0 \quad xy<1$

$x=\frac{1}{2}, y=\frac{1}{3}$

$xy=\frac{1}{6}$

$\tan^{-1}\frac{4n}{(n^2)^2 - 2n^2 + 1 + 1} = \tan^{-1}\frac{4n}{(n^2-1)^2+1} = \tan^{-1}\frac{4n}{1+(n-1)^2(n+1)^2} = \tan^{-1}\frac{4n}{1+(n^2+1-2n)(n^2+1+2n)}$

$= \tan^{-1}\frac{\overset{x}{(n^2+1+2n)} - \overset{y}{(n^2+1-2n)}}{1+\underset{x}{(n^2+1+2n)}\underset{y}{(n^2+1-2n)}}$

$= \tan^{-1}(n^2+1+2n) - \tan^{-1}(n^2+1-2n)$

$\sum_{n=1}^{\infty}\left[\tan^{-1}(n^2+1+2n) - \tan^{-1}(n^2+1-2n)\right]$

$\cancel{\tan^{-1}4} + \boxed{\tan^{-1}0} + \cancel{\tan^{-1}9}\boxed{-\tan^{-1}1} + \tan^{-1}16 - \cancel{\tan^{-1}4} + \tan^{-1}25 - \cancel{\tan^{-1}9}$ - - $\tan^{-1}(n^2+1+2n) - \tan^{-1}(n^2+1-2n)$

Problems

$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$

The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is

(2019 Main, 10 Jan II)

(a) $\frac{23}{22}$
(b) $\frac{21}{19}$
(c) $\frac{19}{21}$
(d) $\frac{22}{23}$

$1+\sum_{p=1}^{n}2p = 1+2\sum_{p=1}^{n}p$

$= 1+2(1+2+3 \ldots n)$

$= 1+\cancel{2}\frac{n(n+1)}{\cancel{2}} = 1+n(n+1)$

$\cot^{-1}(1+n(n+1)) = \tan^{-1}\left[\frac{1}{1+n(n+1)}\right]$

(a) $\sum_{n=1}^{19}\tan^{-1}\frac{1}{1+n(n+1)} = \sum_{n=1}^{19}\tan^{-1}\frac{(n+1)-n}{1+(n+1)n} = \sum_{n=1}^{19}\left[\tan^{-1}(n+1)-\tan^{-1}n\right]$

$= \cancel{\tan^{-1}2} - \tan^{-1}1 + \cancel{\tan^{-1}3} - \cancel{\tan^{-1}2} + \tan^{-1}4 - \cancel{\tan^{-1}3} \ldots \tan^{-1}20 - \cancel{\tan^{-1}19}$

$= \tan^{-1}20 - \tan^{-1}1 = \tan^{-1}\frac{20-1}{1+20\times1} = \tan^{-1}\frac{19}{21} = \cot^{-1}\frac{21}{19}$

$\cot\cot^{-1}\left(\frac{21}{19}\right) = \frac{21}{19}$

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