Trigonometric Ratios

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BASIC TRIGONOMETRIC IDENTITIES :

(a) $\sin^2\theta + \cos^2\theta = 1$; $-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1 \quad \forall \quad \theta \in R$

(b) $\sec^2\theta - \tan^2\theta = 1$; $|\sec\theta| \ge 1 \quad \forall \quad \theta \in R$

(c) $\text{cosec}^2\theta - \cot^2\theta = 1$; $|\text{cosec}\,\theta| \ge 1 \quad \forall \quad \theta \in R$

$\sec^2 100^\circ - \tan^2 100^\circ = 1$

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IMPORTANT T′ RATIOS:

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin\frac{(2n+1)\pi}{2} = (-1)^n$ & $\cos\frac{(2n+1)\pi}{2} = 0$ where $n \in I$

0°, 30°, 45°, 60°, 90°

15°, 75°

(c) $\sin 15°$ or $\sin\frac{\pi}{12} = \boxed{\frac{\sqrt{3}-1}{2\sqrt{2}}} = \cos 75°$ or $\cos\frac{5\pi}{12}$;

$\cos 15°$ or $\cos\frac{\pi}{12} = \boxed{\frac{\sqrt{3}+1}{2\sqrt{2}}} = \sin 75°$ or $\sin\frac{5\pi}{12}$;

$\tan 15° = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$; $\tan 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

(d) $\boxed{\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}}$; $\boxed{\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}}$; $\boxed{\tan\frac{\pi}{8} = \sqrt{2}-1}$ $\boxed{\tan\frac{3\pi}{8} = \sqrt{2}+1}$

(e) $\sin\frac{\pi}{10}$ or $\sin 18° = \frac{\sqrt{5}-1}{4}$ & $\cos 36°$ or $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$

30° $180+30$ $360+30$
$180-30$ $360-30$

Trigonometric Functions Of Allied Angles :

If θ is any angle, then $-\theta$, $90 \pm \theta$, $180 \pm \theta$, $270 \pm \theta$, $360 \pm \theta$ etc. are called **Allied Angles.**

(a) $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$

(b) $\sin(90^\circ - \theta) = \cos\theta$; $\cos(90^\circ - \theta) = \sin\theta$

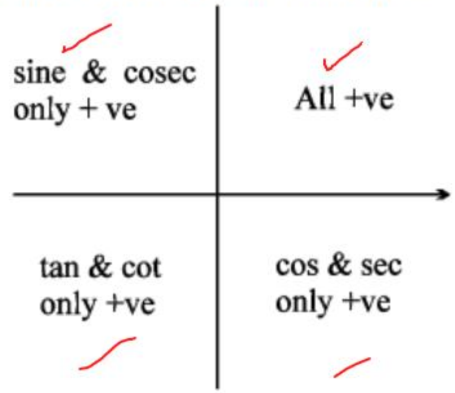
(c) $\sin(90^\circ + \theta) = \cos\theta$; $\cos(90^\circ + \theta) = -\sin\theta$

(d) $\sin(180^\circ - \theta) = \sin\theta$; $\cos(180^\circ - \theta) = -\cos\theta$

(e) $\sin(180^\circ + \theta) = -\sin\theta$; $\cos(180^\circ + \theta) = -\cos\theta$

(f) $\sin(270^\circ - \theta) = -\cos\theta$; $\cos(270^\circ - \theta) = -\sin\theta$

(g) $\sin(270^\circ + \theta) = -\cos\theta$; $\cos(270^\circ + \theta) = \sin\theta$



$\sin 330^\circ = \sin(360-30) = -\sin 30^\circ = -\frac{1}{2}$

$= \sin(270+60^\circ) = -\cos 60^\circ = -\frac{1}{2}$

$a^3+b^3=(a+b)(a^2+b^2-ab)$

Problems

$a^2+b^2=(a+b)^2-2ab$

The expression

$$3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$$

$$-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$$

is equal to

(a) 0 (b) 1

(c) 3 (d) $\sin 4\alpha + \cos 6\alpha$

$3(1-2\cos^2\alpha\sin^2\alpha) - 2(1-3\sin^2\alpha\cos^2\alpha)$

$3-6\cancel{\cos^2\alpha\sin^2\alpha}-2+\cancel{6\cos^2\alpha\sin^2\alpha}=1$

$3[\cos^4\alpha+\sin^4\alpha]-2[\cos^6\alpha+\sin^6\alpha]$

$\cos^4\alpha+\sin^4\alpha=(\cos^2\alpha+\sin^2\alpha)^2-2\cos^2\alpha\sin^2\alpha$

$=1-2\cos^2\alpha\sin^2\alpha$

$\cos^6\alpha+\sin^6\alpha=(\cos^2\alpha+\sin^2\alpha)(\cos^4\alpha+\sin^4\alpha-\cos^2\alpha\sin^2\alpha)$

$=(1-2\sin^2\alpha\cos^2\alpha-\cos^2\alpha\sin^2\alpha)$

$=1-3\cos^2\alpha\sin^2\alpha$

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Problems

$\theta \in 2^{nd}/4^{th}.$

If $\boxed{\tan\theta = -\frac{4}{3}}$, then $\sin\theta$ is

$\frac{4}{3} = \frac{p}{b}$

$h = 5$

$\sin\theta = \pm\frac{4}{5}$

(a) $-\frac{4}{5}$ but not $\frac{4}{5}$

(b) $-\frac{4}{5}$ or $\frac{4}{5}$

(c) $\frac{4}{5}$ but not $-\frac{4}{5}$

(d) None of the above

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Trigonometric Functions Of Sum Or Difference Of Two Angles :

* (a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

* (b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(c) $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A = \sin(A+B) \cdot \sin(A-B)$

(d) $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B) \cdot \cos(A-B)$

* (e) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

* (f) $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

Problems

If $\alpha + \beta = \dfrac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan\alpha$ equals

(a) $2(\tan\beta + \tan\gamma)$ (b) $\tan\beta + \tan\gamma$

(c) $\tan\beta + 2\tan\gamma$ (d) $2\tan\beta + \tan\gamma$

$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

1) $\alpha + \beta = \dfrac{\pi}{2}$

$\alpha = \dfrac{\pi}{2} - \beta$

$\tan\alpha = \tan\left(\dfrac{\pi}{2} - \beta\right)$

$\tan\alpha = \cot\beta$

$\tan\alpha = \dfrac{1}{\tan\beta}$

$\boxed{\tan\alpha \tan\beta = 1}$

$\beta + \gamma = \alpha$

$\gamma = \alpha - \beta$

$\tan\gamma = \tan(\alpha - \beta)$

$\tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$

$\tan\gamma = \dfrac{\tan\alpha - \tan\beta}{1 + 1}$

$2\tan\gamma = \tan\alpha - \tan\beta$

$\tan\alpha = \tan\beta + 2\tan\gamma$

Factorisation Of The Sum Or Difference Of Two sines Or cosines :

(a) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(b) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(c) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(d) $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

Transformation Of Products Into Sum Or Difference Of sines & cosines :

(a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(b) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(d) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Problems

$1+\cos 2\theta = 2\cos^2\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cdot\cos\left(\frac{C-D}{2}\right)$

The value of $\cos^2 10° - \cos 10° \cos 50° + \cos^2 50°$ is

2019 Main

(a) $\frac{3}{2}(1+\cos 20°)$ (b) $\frac{3}{4} + \cos 20°$

(c) $3/2$ (d) $3/4$

$E = \frac{1}{2}\left[2\cos^2 10° - 2\cos 10°\cos 50° + 2\cos^2 50°\right]$

$= \frac{1}{2}\left[1+\cos 20° - [\cos(60) + \cos 40°] + 1+\cos 100°\right]$

$= \frac{1}{2}\left[2 + \cos 20° - \frac{1}{2} - \cos 40° + \cos 100°\right]$

$= \frac{1}{2}\left[\frac{3}{2} + \cos 20° - \cos 40° + \cos 100°\right] = \frac{3}{4} + \frac{1}{2}\left[\underbrace{\cos 20° - \cos 40° + \cos 100°}\right]$

$= \frac{3}{4} + \frac{1}{2}\left[\cancel{2\cos 60°}\cdot\cos 40° - \cos 40°\right]$

$= \frac{3}{4} + \frac{1}{2}\left(\cos 40° - \cos 40°\right)$

$= 3/4.$

Multiple Angles And Half Angles :

$\sin 2A = 2\sin A\cos A = \dfrac{2\tan A}{1+\tan^2 A}$

(a) $\sin 2A = 2\sin A\cos A$; $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

(b) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$;

$\cos\theta = \cos^2\dfrac{\theta}{2} - \sin^2\dfrac{\theta}{2} = 2\cos^2\dfrac{\theta}{2} - 1 = 1 - 2\sin^2\dfrac{\theta}{2}$.

$\boxed{2\cos^2 A = 1 + \cos 2A}$, $\boxed{2\sin^2 A = 1 - \cos 2A}$; $\tan^2 A = \dfrac{1-\cos 2A}{1+\cos 2A}$

$2\cos^2\dfrac{\theta}{2} = 1 + \cos\theta$, $2\sin^2\dfrac{\theta}{2} = 1 - \cos\theta$.

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$= \dfrac{1-\tan^2 A}{1+\tan^2 A}$

$\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

(c) $\boxed{\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}}$; $\tan\theta = \dfrac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$

(d) $\boxed{\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}}$, $\boxed{\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}}$

(e) $\sin 3A = 3\sin A - 4\sin^3 A$

(f) $\cos 3A = 4\cos^3 A - 3\cos A$

(g) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

Problems

If $\cos(\alpha+\beta)=\frac{3}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$ and $0<\alpha,\beta<\frac{\pi}{4}$, then $\tan(2\alpha)$ is equal to **(2019 Main, 8 April I)**

(a) $\frac{63}{52}$ (b) $\frac{63}{16}$ (c) $\frac{21}{16}$ (d) $\frac{33}{52}$

$0<\alpha<\frac{\pi}{4}$

$0<\beta<\frac{\pi}{4}$

$0<\alpha+\beta<\frac{\pi}{2}$

$0-\frac{\pi}{4}<\alpha-\beta<\frac{\pi}{4}-0$

Max Min

$\boxed{-\frac{\pi}{4}<\alpha-\beta<\frac{\pi}{4}}$

$\cos(\alpha+\beta)=\frac{3}{5}$, $\sin(\alpha-\beta)=\frac{5}{13}$

$\cos(\alpha+\beta)=\frac{3}{5} \Rightarrow \tan(\alpha+\beta)=\frac{4}{3}$

$\sin(\alpha-\beta)=\frac{5}{13} \Rightarrow \tan(\alpha-\beta)=\frac{5}{12}$

$\tan 2\alpha = \tan(\underbrace{\alpha+\beta}_{A}+\underbrace{\alpha-\beta}_{B})$

$\tan 2\alpha = \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\cdot\tan(\alpha-\beta)} = \frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\times\frac{5}{12}} = \frac{48+15}{36-20} = \frac{63}{16}$

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Problems

$\sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta$

$\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$

Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ for $k = 1, 2, 3 \ldots$. Then, for all $x \in R$, the value of $f_4(x) - f_6(x)$ is equal to

(2019 Main, 11 Jan I)

(a) $\frac{1}{12}$ (b) $\frac{5}{12}$ (c) $\frac{-1}{12}$ (d) $\frac{1}{4}$

$f_4(x) - f_6(x)$

$\frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\sin^6 x + \cos^6 x)$

$= \frac{1}{4}(1 - 2\sin^2 x\cos^2 x) - \frac{1}{6}(1 - 3\sin^2 x\cos^2 x)$

$= \frac{1}{4} - \frac{1}{6} - \cancel{\frac{1}{2}\sin^2 x\cos^2 x} + \cancel{\frac{1}{2}\sin^2 x\cos^2 x}$

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Note :

$$\sin A \ \sin(60^\circ - A) \ \sin(60^\circ + A) = \frac{1}{4}\sin 3A$$

$$\cos A \ \cos(60^\circ - A) \ \cos(60^\circ + A) = \frac{1}{4}\cos 3A$$

$$\tan A \ \tan(60^\circ - A) \ \tan(60^\circ + A) = \tan 3A$$

The value of $\sin 20^\circ \sin 40^\circ \sin 60^\circ \sin 80^\circ$ is -

(A) $\frac{3}{8}$ (B) $\frac{1}{8}$ (C) $\frac{3}{16}$ (D) None of these

$\sin 60^\circ [\sin 20^\circ \sin 40^\circ \sin 80^\circ]$

$\sin 60^\circ [\sin 20^\circ \cdot \sin(60-20) \cdot \sin(60+20^\circ)]$

$\sin 60^\circ \times \frac{1}{4}\sin(3\times 20^\circ) = \frac{1}{4}\sin^2 60^\circ = \frac{1}{4}\times\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{16}$

Problems

The value of $\sin 10^\circ \sin 30^\circ \sin 50^\circ \sin 70^\circ$ is

(2019 Main, 9 April II)

(a) $\frac{1}{36}$ (b) $\frac{1}{32}$ (c) $\frac{1}{16}$ (d) $\frac{1}{18}$

$\sin 30^\circ \left[\sin 10^\circ \cdot \sin(60-10) \cdot \sin(60+10) \right]$

$\sin 30^\circ \times \frac{1}{4} \sin(3\times 10^\circ)$

$\frac{1}{4} \sin^2 30^\circ = \frac{1}{4} \times \left(\frac{1}{2}\right)^2 = \frac{1}{16}$

Continued product of cosine Series :

$$2^n \sin A \cos A \cos 2A \cos 4A \cos 8A \ldots \cos(2^{n-1}A) = \frac{1}{2^n \sin A}\sin(2^n A)$$

$2^n \sin\theta$

$2\sin 2\theta \to \sin 4\theta \to \sin 8\theta \to \sin 16\theta \to \sin 32\theta \to \sin 64\theta$

$$\boxed{2\sin\theta\cos\theta} \cdot \cos 2\theta \cdot \cos 4\theta \cdot \cos 8\theta \cdot \cos 16\theta \cdot \cos 32\theta$$

$2^6 \sin\theta$

$$= \frac{\sin 64\theta}{2^6 \sin\theta}$$

Problems

The value of

$\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\ldots\ldots\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is (2019 Main, 10 Jan II)

(a) $\frac{1}{1024}$ (b) $\frac{1}{2}$ (c) $\frac{1}{512}$ (d) $\frac{1}{256}$

9 terms

$2\sin\frac{\pi}{2^2}$

$$\frac{2^9}{2^9}\sin\frac{\pi}{2^{10}}\times\left[\cos\frac{\pi}{2^{10}}\times\cos\frac{\pi}{2^9}\times\cos\frac{\pi}{2^8}\times\cos\frac{\pi}{2^7}\ldots\ldots\cos\frac{\pi}{2^2}\right]$$

$$\frac{1}{2^9}\left[\sin\frac{2\pi}{2^2}\right]=\frac{1}{2^9}\sin\frac{\pi}{2}=\frac{1}{2^9}=\frac{1}{512}$$

THREE ANGLES:

(a) $\tan(A+B+C) = \dfrac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$

NOTE IF : **(i)** $A+B+C=\pi$ then $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(ii) $A+B+C=\dfrac{\pi}{2}$ then $\tan A\tan B+\tan B\tan C+\tan C\tan A=1$

(b) If $\boxed{A+B+C=\pi}$ then : **(i)** $\sin 2A+\sin 2B+\sin 2C = 4\sin A\sin B\sin C$

(ii) $\sin A+\sin B+\sin C = 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

Conditional identity

$A+B+C=\pi.$

Problems

If $\alpha + \beta + \gamma = 2\pi$, then

(a) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(b) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} + \tan\frac{\beta}{2}\tan\frac{\gamma}{2} + \tan\frac{\gamma}{2}\tan\frac{\alpha}{2} = 1$

(c) $\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = -\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(d) None of the above

$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi$

$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \tan\frac{\alpha}{2} \times \tan\frac{\beta}{2} \times \tan\frac{\gamma}{2}$

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Problems

The maximum value of

$$3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$$

for any real value of θ is (2019 Main, 12 Jan I)

(a) $\frac{\sqrt{79}}{2}$ (b) $\sqrt{34}$

(c) $\sqrt{31}$ (d) $\sqrt{19}$

$y = a\cos\theta + b\sin\theta$

$y_{max} = \sqrt{a^2+b^2}$

$y_{min} = -\sqrt{a^2+b^2}$.

$y = 3\cos\theta + 4\sin\theta$

Range $= [-5, 5]$

$3\cos\theta + 5\left[\sin\theta \cdot \cos\frac{\pi}{6} - \cos\theta \cdot \sin\frac{\pi}{6}\right]$

$3\cos\theta + 5\left[\frac{\sqrt{3}\sin\theta}{2} - \frac{\cos\theta}{2}\right]$

$\left(3 - \frac{5}{2}\right)\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta$

$\frac{1}{2}\cos\theta + \frac{5\sqrt{3}}{2}\sin\theta$ $\sqrt{a^2+b^2}$

$\sqrt{\frac{1}{4} + \frac{75}{4}} = \sqrt{\frac{76}{4}} = \sqrt{19}$