Sequence & Series

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Sequence

DEFINITION :

A sequence is a set of terms in a definite order with a rule for obtaining the terms.
e.g. $1, 1/2, 1/3, \ldots\ldots, 1/n, \ldots\ldots$ is a sequence.

A.P

G.P

H.P

A.G.P

ARITHMETIC PROGRESSION (AP) :

AP is a sequence whose terms increase or decrease by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then AP can be written as $a, a+d, a+2d, \ldots\ldots\ a+(n-1)d, \ldots\ldots$

$a, a+d, a+2d$ - - —

✓ n^{th} term of this AP $t_n = a + (n-1)d$, where $d = a_n - a_{n-1}$.

The sum of the first n terms of the AP is given by ; $S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a + l]$.

where l is the last term.

$a_n = a + (n-1)d$

$S_n = \frac{n}{2}(2a + (n-1)d)$ or $\frac{n}{2}(a + l)$

Problems

If 19th term of a non-zero AP is zero, then its (49th term) : (29th term) is (2019 Main, 11 Jan II)

(a) 1 : 3 (b) 4 : 1

(c) 2 : 1 (d) 3 : 1

$a, a+d, a+2d$ ----

$a_n = a+(n-1)d$

$a_{19} = a+18d = 0 \Rightarrow \boxed{a = -18d}$

$\frac{a_{49}}{a_{29}} = \frac{a+48d}{a+28d} = \frac{-18d+48d}{-18d+28d} = \frac{30d}{10d} = 3:1$

Problems

Let S_n denote the sum of the first n terms of an AP. If $S_4 = 16$ and $S_6 = -48$, then S_{10} is equal to

(2019 Main, 12 April I)

(a) -260 (b) -410 (c) -320 (d) -380

$S_4 = 16$

$S_{10} = ?$

$S_6 = -48$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$S_4 = \frac{4}{2}(2a + 3d)$

$16 = 2(2a + 3d)$

$\boxed{2a + 3d = 8}$ — 1

$S_6 = -48$

$-48 = \frac{6}{2}(2a + 5d)$

$\boxed{2a + 5d = -16}$ — 2

$d = -12$, $a = 22$

$S_{10} = \frac{10}{2}(2 \times 22 + 9(-12))$

$= \frac{10}{2}(44 - 108)$

$= -\frac{10}{2} \times 64 = -320.$

$a,\ a+d,\ a+2d$ — —

NOTES :

(i) If each term of an A.P. is increased, decreased, multiplied or divided by the same non zero number, then the resulting sequence is also an AP.

(ii) Three numbers in AP can be taken as $a-d,\ a,\ a+d$; four numbers in AP can be taken as $a-3d,\ a-d,\ a+d,\ a+3d$; five numbers in AP are $a-2d,\ a-d,\ a,\ a+d,\ a+2d$ & six terms in AP are $a-5d,\ a-3d,\ a-d,\ a+d,\ a+3d,\ a+5d$ etc.

(iii) The common difference can be zero, positive or negative.

(iv) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it.

(vi) $\boxed{t_r = S_r - S_{r-1}}$

(vii) If a, b, c are in AP $\Rightarrow 2b = a + c$.

$S_n = a_1 + a_2 + a_3 - - a_n$

$S_{n-1} = a_1 + a_2 + a_3 - a_{n-1}$

$S_n - S_{n-1} = a_n$

$\boxed{a_n = S_n - S_{n-1}}$

$a_1 + a_2 + a_3 + a_4 \cdot - - - a_{n-3} + a_{n-2} + a_{n-1} + a_n$

$a + a+d + a+2d \qquad l-2d + l-d + l$

$a+l$.

$a+l$

Problems

If $a_1, a_2, a_3, \ldots, a_n$ are in AP and $a_1 + a_4 + a_7 + \ldots + a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to

(2019 Main, 10 April I)

(a) 64 (b) 76

(c) 98 (d) 38

$$a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 114$$

$a+l$, $a+l$, $a+l$

$$3(a+l) = 114$$

$$a+l = 38$$

$$a_1 + a_6 + a_{11} + a_{16} = 2(a+l)$$

$a+l$, $a+l$

$$= 2 \times 38$$

$$= 76.$$

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Problems

If the sum and product of the first three terms in an AP are 33 and 1155, respectively, then a value of its 11th term is (2019 Main, 9 April II)

(a) 25 (b) –36

(c) –25 (d) –35

3 terms in an A.P

$a-d, a, a+d$

$a-d+a+a+d = 33$

$3a = 33$

$\boxed{a = 11}$

$(a-d)\,a\,(a+d) = 1155$

$(11-d)(11+d)\,11 = 1155$

$11^2 - d^2 = 105$

$121 - d^2 = 105$

$d^2 = 16$

$d = \pm 4$

$a = 11, \; d = \pm 4$

$\boxed{d = 4 \quad 7, 11, 15}$

$\boxed{d = -4 \quad 15, 11, 7}$

$a_{11} = A + 10D$

$= 7 + 10(4)$

$= 47$ ✗

$a_{11} = 15 + 10(-4)$

$= 15 - 40$

$= -25$

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GEOMETRIC PROGRESSION (GP) :

GP is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the proceeding terms multiplied by a constant . Thus in a GP the ratio of successive terms is constant. This constant factor is called the **COMMON RATIO** of the series & is obtained by dividing any term by that which immediately proceeds it. Therefore a, ar, ar^2, ar^3, ar^4, ... is a GP with a as the first term & r as common ratio.

a, ar, ar^2, ar^3 —— ar^{n-1}

(i) n^{th} term $= a\,r^{n-1}$

(ii) Sum of the Ist n terms i.e. $S_n = \dfrac{a\left(r^n - 1\right)}{r-1}$, if $r \neq 1$.

(iii) Sum of an infinite GP when $|r| < 1$ when $n \to \infty$ $r^n \to 0$ if $|r| < 1$ therefore,

$$S_\infty = \frac{a}{1-r} (|r| < 1) \; .$$ $-1 < r < 1$

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2, 6, 18, 54 - —

Multiply by 5 10, 30, 90, 270 - -

(iv) If each term of a GP be multiplied or divided by the same non-zero quantity, the resulting sequence is also a GP.

(v) Any 3 consecutive terms of a GP can be taken as $a/r, a, ar$; any 4 consecutive terms of a GP can be taken as $a/r^3, a/r, ar, ar^3$ & so on.

(vi) If a, b, c are in GP $\Rightarrow b^2 = ac$.

$$\frac{b}{a} = \frac{c}{b} \Rightarrow \boxed{b^2 = ac}$$

Problems

Let $a_1, a_2, \ldots, a_{10}$ be a GP. If $\frac{a_3}{a_1} = 25$, then $\frac{a_9}{a_5}$ equals (2019 Main,

(a) 5^3 (b) $2(5^2)$ (c) $4(5^2)$ (d) 5^4

$a_n = ar^{n-1}$

$a_1 = a$, $a_3 = ar^2$

$a_9 = ar^8$, $a_5 = ar^4$

$\frac{a_3}{a_1} = 25 \Rightarrow \frac{\cancel{a}r^2}{\cancel{a}} = 25$

$r = \pm 5$

$\frac{a_9}{a_5} = \frac{\cancel{a}r^8}{\cancel{a}r^4} = r^4$

$= (\pm 5)^4$

$= 5^4.$

Problems

The product of three consecutive terms of a GP is 512. If 4 is added to each of the first and the second of these terms, the three terms now form an AP. Then, the sum of the original three terms of the given GP is

(2019 Main, 12 Jan I)

(a) 36 (b) 28 (c) 32 (d) 24

$\frac{a}{r} \times a \times ar = 512$

$a^3 = 512$

$\boxed{a = 8}$

$\frac{a}{r}+4, \ a+4, \ ar$ in A.P

$2(a+4) = \frac{a}{r}+4+ar.$

$2 \times 12 = \frac{8}{r}+4+8r.$

$20 = 8\left(r+\frac{1}{r}\right)$

$5 = 2\left(r+\frac{1}{r}\right)$

$5 = 2\left(\frac{r^2+1}{r}\right)$

$\cancel{2r^2-11r+2=0}$

$2r^2-5r+2=0$

$r = 2 \ \& \ \frac{1}{2}, \ a = 8$

4, 8, 16   16, 8, 4

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HARMONIC PROGRESSION (HP) :

A sequence is said to HP if the reciprocals of its terms are in AP.

If the sequence $a_1, a_2, a_3, \ldots, a_n$ is an HP then $1/a_1, 1/a_2, \ldots, 1/a_n$ is an AP & converse. Here we do not have the formula for the sum of the n terms of an HP. For HP whose first term is a & second term is b, the n^{th} term is $t_n = \dfrac{ab}{b+(n-1)(a-b)}$.

If a, b, c are in HP $\Rightarrow b = \dfrac{2ac}{a+c}$ or $\dfrac{a}{c} = \dfrac{a-b}{b-c}$.

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ in A.P

$\frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \frac{2}{b} = \frac{a+c}{ac} \Rightarrow \boxed{b = \frac{2ac}{a+c}}$ condition for HP

Problems

If $a_1, a_2, a_3, \ldots$ are in a harmonic progression with $a_1 = 5$ and $a_{20} = 25$. Then, the least positive integer n for which $a_n < 0$, is **(2012)**

(a) 22 (b) 23 (c) 24 (d) 25

$a_1, a_2, a_3 - -$ H.P

$a_1 = 5$, $a_{20} = 25$

$a_n < 0$

$t_1, t_2, t_3 - -$ A.P

$t_1 = \frac{1}{5}$, $t_{20} = \frac{1}{25}$

$t_n < 0$

$t_n = t_1 + (n-1)d$

$= \frac{1}{5} + (n-1)\left(\frac{-4}{25 \times 19}\right)$

$= \frac{95 - 4n + 4}{25 \times 19}$

$t_{20} = t_1 + 19d$

$\frac{1}{25} = \frac{1}{5} + 19d$

$\frac{1}{25} - \frac{1}{5} = 19d$

$\frac{1-5}{25} = 19d$

$d = \frac{-4}{25 \times 19}$

$t_n = \frac{99 - 4n}{25 \times 19}$

$t_n < 0$

$99 - 4n < 0$

$99 < 4n$

$n > \frac{99}{4}$

$n > 24.75$

$n = 25$

ARITHMETIC MEAN :

If three terms are in AP then the middle term is called the AM between the other two, so if a, b, c are in AP, b is AM of a & c .

AM for any n positive number $a_1, a_2, \ldots, a_n$ is ; $A = \frac{a_1 + a_2 + a_3 + \ldots\ldots + a_n}{n}$.

$a_1, a_2, a_3 \text{ --- } a_n$

A.M. G.M H.M

$\frac{a_1 + a_2 \text{ --- } a_n}{n}$ $\quad (a_1 \times a_2 \times a_3 \text{ --- } a_n)^{1/n}$ $\quad \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \text{ --- } \frac{1}{a_n}}$

If numbers are +ve $\quad A.M \geq G.M \geq H.M$.

n - ARITHMETIC MEANS BETWEEN TWO NUMBERS :

If a, b are any two given numbers & $a, A_1, A_2, \ldots, A_n, b$ are in AP then $A_1, A_2, \ldots A_n$ are the n AM's between a & b .

$$A_1 = a + \frac{b-a}{n+1}, \; A_2 = a + \frac{2(b-a)}{n+1}, \; \ldots\ldots, A_n = a + \frac{n(b-a)}{n+1}$$

$$= a + d, \qquad = a + 2d, \; \ldots\ldots, A_n = a + nd, \text{ where } d = \frac{b-a}{n+1}$$

Note : Sum of n AM's inserted between a & b is equal to n times the single AM between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single AM between a & b.

$\overbrace{a, A_1, A_2, A_3, A_4, A_5, b}^{A.P}$

$\overbrace{a, G_1, G_2, G_3, G_4, G_5, b}^{G.P}$

GEOMETRIC MEANS :

If a, b, c are in GP, b is the GM between a & c.

$b^2 = ac$, therefore $b = \sqrt{ac}$; $a > 0$, $c > 0$.

n-GEOMETRIC MEANS BETWEEN a, b :

If a, b are two given numbers & a, $G_1, G_2, \ldots, G_n$, b are in GP. Then $G_1, G_2, G_3, \ldots, G_n$ are n GMs between a & b.

$G_1 = a(b/a)^{1/n+1}$, $G_2 = a(b/a)^{2/n+1}$, ... , $G_n = a(b/a)^{n/n+1}$

$= ar$, $= ar^2$, ... $= ar^n$, where $r = (b/a)^{1/n+1}$

Note : The product of n GMs between a & b is equal to the n^{th} power of the single GM between a & b

i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single GM between a & b.

HARMONIC MEAN :

If a, b, c are in HP, b is the HM between a & c, then $b = 2ac/[a+c]$.

THEOREM :

If A, G, H are respectively AM, GM, HM between a & b both being unequal & positive then,

(i) $G^2 = AH$

(ii) $A > G > H$ $(G > 0)$. Note that A, G, H constitute a GP.

Problems

If m is the AM of two distinct real numbers l and $n (l, n > 1)$ and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals **(2015)**

(a) $4l^2mn$ (b) $4lm^2n$ (c) lmn^2 (d) $l^2m^2n^2$

$l, \quad n$

$m = \frac{l+n}{2}$

$l + n = 2m$

l, G_1, G_2, G_3, n

l, lr, lr^2, lr^3, n

$n = lr^4$

$\boxed{r^4 = \frac{n}{l}}$

$G_1^4 + 2G_2^4 + G_3^4$

$= l^4r^4 + 2l^4r^8 + l^4r^{12}$

$= l^4(r^4 + 2r^8 + r^{12})$

$= l^4\left[\frac{n}{l} + 2\frac{n^2}{l^2} + \frac{n^3}{l^3}\right]$

$= l^4\left[\frac{nl^2 + 2n^2l + n^3}{l^3}\right] = ln[l^2 + 2ln + n^2]$

$= ln[l+n]^2$

$= ln(2m)^2 = 4m^2ln$

ARITHMETICO-GEOMETRIC SERIES :

A series each term of which is formed by multiplying the corresponding term of an AP & GP is called the **Arithmetico-Geometric Series**. e.g. $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots$

Here 1, 3, 5, ... are in AP & 1, x, x^2, x^3 ..... are in GP.

Standart appearance of an Arithmetico-Geometric Series is

Let $S_n = a + (a + d)r + (a + 2d)r^2 + \ldots\ldots + [a + (n-1)d]\ r^{n-1}$

$S_n = \boxed{1} + \boxed{3}x^1 + \boxed{5}x^2 + \boxed{7}x^3$ ———

A.P. × G.P = A.G.P.

Problems

The sum $\sum_{k=1}^{20} k \frac{1}{2^k}$ is equal to (2019 Main, 8 April II)

(a) $2-\frac{11}{2^{19}}$ (b) $1-\frac{11}{2^{20}}$

(c) $2-\frac{3}{2^{17}}$ (d) $2-\frac{21}{2^{20}}$

$S = 1\times\frac{1}{2} + 2\times\frac{1}{2^2} + 3\times\frac{1}{2^3} - - - - 20\times\frac{1}{2^{20}}$

$\frac{1}{2}\times S = 1\times\frac{1}{2^2} + 2\times\frac{1}{2^3} \quad 19\times\frac{1}{2^{20}} + 20\times\frac{1}{2^{21}}$

$\frac{S}{2} = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} - - - - \frac{1}{2^{20}}\right) - \frac{20}{2^{21}}$

$S = \left(1 + \frac{1}{2} + \frac{1}{2^2} - - - \frac{1}{2^{19}}\right) - \frac{20}{2^{20}}$

$S = \frac{1\left(1-\left(\frac{1}{2}\right)^{19}\right)}{1/2} - \frac{20}{2^{20}}$

SIGMA NOTATIONS

THEOREMS :

(i) $\sum\limits_{r=1}^{n} (a_r \pm b_r) = \sum\limits_{r=1}^{n} a_r \pm \sum\limits_{r=1}^{n} b_r.$

(ii) $\sum\limits_{r=1}^{n} k\, a_r = k \sum\limits_{r=1}^{n} a_r.$

(iii) $\sum\limits_{r=1}^{n} k = nk$; where k is a constant.

RESULTS

(i) $\sum\limits_{r=1}^{n} r = \frac{n\,(n+1)}{2}$ (sum of the first n natural nos.)

$1+2+3 \ldots - n = \frac{n(n+1)}{2}$

(ii) $\sum\limits_{r=1}^{n} r^2 = \frac{n\,(n+1)\,(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

$1^2+2^2+3^2 - - n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum\limits_{r=1}^{n} r^3 = \frac{n^2\,(n+1)^2}{4} \left[\sum\limits_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

$1^3+2^3 - - n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Problems

The sum of the series $1 + 2 \times 3 + 3 \times 5 + 4 \times 7 + \ldots$ upto 11th term is (2019 Main, 9 April II)

(a) 915 (b) 946 (c) 916 (d) 945

~~S = 1×1+~~

$S = 1 + 2\times3 + 3\times5 + 4\times7 \ \text{---}$ 11 terms.

$S = 1\times1 + 2\times3 + 3\times5 + 4\times7 \ \text{---}$

$a_n = n(2n-1) = 2n^2 - n.$

$S_n = \Sigma a_n = \Sigma(2n^2 - n)$

$S_n = \Sigma 2n^2 - \Sigma n$

$S_n = 2\Sigma n^2 - \Sigma n$

$= 2 \times \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$

$S_n = \frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2}$

$S_{11} = \frac{11 \times \cancel{12}^{4} \times 23}{\cancel{3}} - \frac{11 \times \cancel{12}^{6}}{\cancel{2}}$

$= 44 \times 23 - 66.$

$= 946.$

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Problems

The sum of series $1 + \frac{1^3 + 2^3}{1 + 2} + \frac{1^3 + 2^3 + 3^3}{1 + 2 + 3} + \ldots + \frac{1^3 + 2^3 + 3^3 + \ldots + 15^3}{1 + 2 + 3 + \ldots + 15} - \boxed{\frac{1}{2}(1 + 2 + 3 + \ldots + 15)} \to 60$ is equal to (2019 Main, 10 April II)

(a) 620 (b) 660 (c) 1240 (d) 1860

$$\underbrace{\frac{1^3}{1} + \frac{1^3+2^3}{1+2} + \frac{1^3+2^3+3^3}{1+2+3} \ldots \frac{1^3+2^3 \ldots 15^3}{1+2 \ldots 15}}_{15 \text{ terms.}} - \frac{1}{2}(1+2+3 \ldots 15)$$

$$\frac{1}{2} \times \frac{15 \times 16}{2}$$

$$a_n = \frac{1^3+2^3 \ldots n^3}{1+2+3 \ldots n} = \frac{\left(\frac{n(n+1)}{2}\right)^2}{\frac{n(n+1)}{2}} = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}$$

$$- 60$$

$$S_{15} = \Sigma \frac{n^2}{2} + \Sigma \frac{n}{2} = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$- 60$$