Quadratic Polynomials

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Quadratic Equation

The general form of a quadratic equation in x is, $ax^2+bx+c=0$, where $a, b, c \in R$ & $a \neq 0$.

leading coeff.

1. The solution of the quadratic equation, $ax^2+bx+c=0$ is given by $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

   The expression $b^2-4ac=D$ is called the discriminant of the quadratic equation.

2. If α & β are the roots of the quadratic equation $ax^2+bx+c=0$, then;

   (i) $\alpha+\beta=-b/a$ (ii) $\alpha\beta=c/a$ (iii) $\alpha-\beta=\sqrt{D}/a$.

$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$= \frac{b^2}{a^2}-\frac{4c}{a} = \frac{b^2-4ac}{a^2}$.

$(\alpha-\beta)^2=\frac{D}{a^2} \Rightarrow |\alpha-\beta|=\frac{\sqrt{D}}{a}$

A quadratic equation whose roots are α & β is $(x-\alpha)(x-\beta)=0$ i.e. $x^2-(\alpha+\beta)x+\alpha\beta=0$ i.e. $x^2-(\text{sum of roots})x+\text{product of roots}=0$.

Problems

If α and β are the roots of the equation $x^2 - 2x + 2 = 0$, then the least value of n for which $\left(\frac{\alpha}{\beta}\right)^n = 1$ is

(a) 2 (b) 5 **(2019 Main, 8 April I)**

✓~~(c)~~ 4 (d) 3

$x^2 - 2x + 2 = 0 \quad \alpha, \beta$

$\alpha, \beta = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2}$

$= \frac{2 \pm 2i}{2}$

$= 1+i, 1-i$

$\left(\frac{\alpha}{\beta}\right)^n = \left(\frac{1+i}{1-i}\right)^n = 1$

$\frac{1+i}{1-i} \times \frac{(1+i)}{(1+i)} = \frac{1+i^2+2i}{1+1} = i$

$= i^n = 1$

NATURE OF ROOTS:

(A) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in R$ & $a \neq 0$ then ;

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal).

(iii) $D < 0 \Leftrightarrow$ roots are imaginary .

(iv) If $p + iq$ is one root of a quadratic equation, then the other must be the conjugate $p - iq$ & vice versa. ($p, q \in R$ & $i = \sqrt{-1}$).

→ imaginary roots occur in conjugate pairs
$p+iq$ & $p-iq$

$ax^2+bx+c=0$ (exactly two roots)

If $ax^2+bx+c=0$ is an identity then
$a=b=c=0$

$0=0$ (all values will satisfy)
(infinite soln)

Remember that a quadratic equation cannot have three different roots & if it has, it becomes an identity.

Problems

The number of integral values of m for which the equation $(1+m^2)x^2 - 2(1+3m)x + (1+8m) = 0$, has no real root is **(2019 Main, 8 April II)**

(a) 3 (b) infinitely many

(c) 1 (d) 2

$(1+m^2)x^2 - 2(1+3m)x + 1+8m = 0$ No real root

$D < 0$

$4(1+3m)^2 - 4(1+8m)(1+m^2) < 0$

$\cancel{1} + 9m^2 + 6m - [\cancel{1} + 8m + m^2 + 8m^3] < 0$

$8m^2 - 2m - 8m^3 < 0$

$-2m(4m^2 - 4m + 1) < 0$

$\boxed{m(2m-1)^2 > 0} \Rightarrow m > 0$

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Problems

1. If $2 + i\sqrt{3}$ is a root of the equation $x^2 + px + q = 0$, where p and q are real, then $(p, q) = (\ldots,\ldots)$.
2. The coefficient of x^{99} in the polynomial $(x-1)(x-2)\ldots(x-100)$ is....

$(x-1)(x-2)(x-3) = -3x^2 - 2x^2 - 1x^2$

$= x^2(-3-2-1)$

1) $x^2 + px + q = 0$ → $2 + i\sqrt{3}$, $2 - i\sqrt{3}$

$\alpha + \beta = 4 = -p$

$\alpha\beta = (2+i\sqrt{3})(2-i\sqrt{3}) = 4 + 3 = 7 = q$

$\Rightarrow p = -4, q = 7$

2) $(x-1)(x-2)(x-3) \ldots (x-100)$

$x^{99}(-1-2-3-4-\ldots-100)$

$x^{99}(-1)(1+2+3\ldots100)$

$x^{99}(-1)\frac{100\times101}{2}$

$\boxed{-5050}x^{99}$

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Rational No

(B) Consider the quadratic equation $ax^2 + bx + c = 0$ where $a, b, c \in Q$ & $a \neq 0$ then;

(i) If $D > 0$ & is a perfect square, then roots are rational & unequal.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is a surd) then the other root must be the conjugate of it i.e. $\beta = p - \sqrt{q}$ & vice versa.

$x = \frac{-b \pm \sqrt{D}}{2a}$ D is a perfect square & a, b, c are rational then roots will be rational.

If coeff are rational, then $p+\sqrt{q}$ & $p-\sqrt{q}$ occur in conjugate pairs.

$2+\sqrt{3}$, $2-\sqrt{3}$

Remember that a quadratic equation cannot have three different roots & if it has, it becomes an identity.

Problems

Let $p, q \in R$. If $2-\sqrt{3}$ is a root of the quadratic $x^2 + px + q = 0$, then (2019 Main)

(a) $q^2 - 4p - 16 = 0$

(b) $p^2 - 4q - 12 = 0$

(c) $p^2 - 4q + 12 = 0$

(d) $q^2 + 4p + 14 = 0$

$S = 2-\sqrt{3} + 2+\sqrt{3} = 4 = -p \Rightarrow p = -4$

$P = (2-\sqrt{3})(2+\sqrt{3}) = 1 = q \Rightarrow q = 1$

$p^2 - 4q - 12 = 16 - 4 - 12 = 0$

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Problems

The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2 - 11x + \alpha = 0$ are rational numbers is $\rightarrow 1, 2, 3, 4 \ldots \infty$

(2019 Main, 9 Jan II)

D should be perfect square

(a) 5
(b) 2
(c) 4
(d) 3 ✓

$D = 121 - 24\alpha$

$D(\alpha=1) = 121 - 24 = 97$ ✗

$D(\alpha=2) = 121 - 48 = 73$ ✗

$D(\alpha=3) = 49$ ✓

$D(\alpha=4) = 25$ ✓

$D(\alpha=5) = 1$ ✓

$D(\alpha=6) = -ve$

$\downarrow$ $-ve$

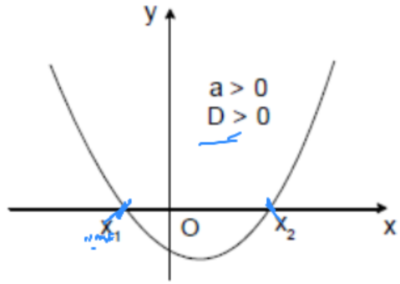
Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a, b, c \in R$ then

$y = -x^2 + 3x - 2$

$a > 0$ concave upwards

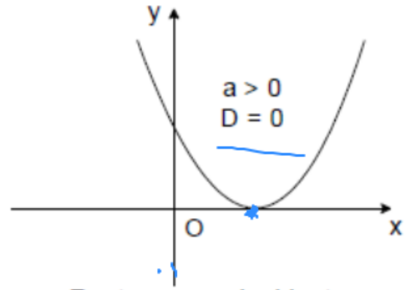
$a < 0$ concave downwards

Fig. 1



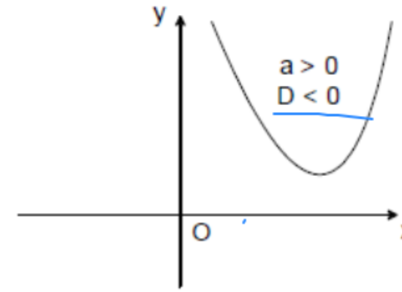
Roots are real & distinct

Fig. 2



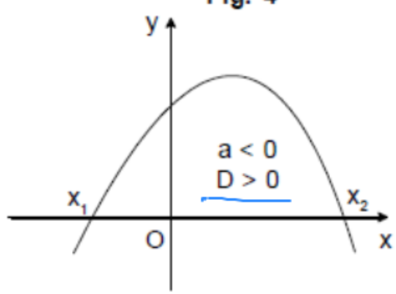
Roots are coincident

Fig. 3



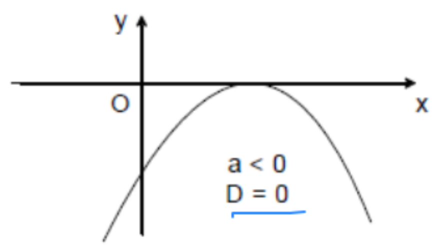
Roots are complex conjugate

Fig. 4



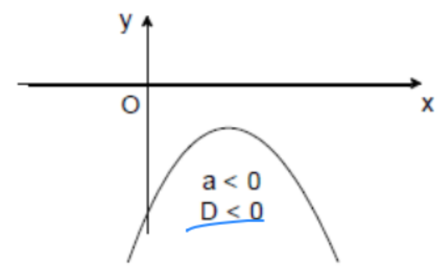
Roots are real & distinct

Fig. 5



Roots are coincident

Fig. 6



Roots are complex conjugate

Consider the quadratic expression, $\boxed{y = ax^2 + bx + c}$, $a \neq 0$ & $a, b, c \in R$ then

$y = -x^2 + 3x - 2$

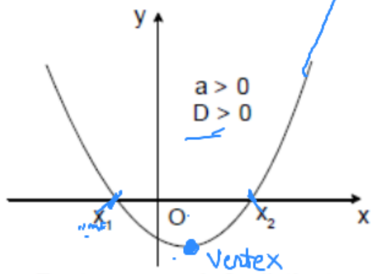
$a > 0$ concave upwards

$a < 0$ concave downwards

Vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$

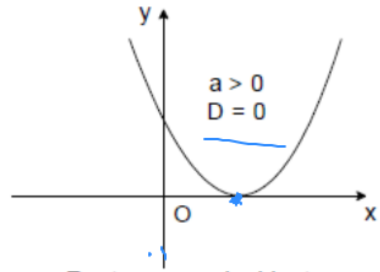
$\left[\frac{-D}{4a}, \infty\right)$

Fig. 1



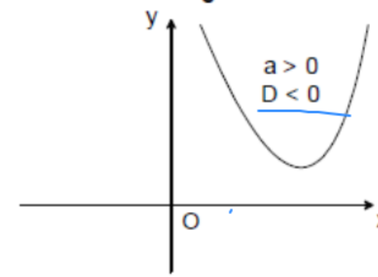
Roots are real & distinct

Fig. 2



Roots are coincident

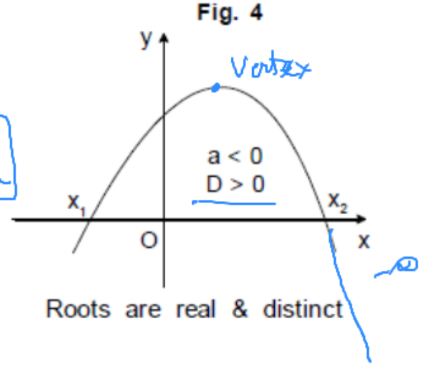
Fig. 3



Roots are complex conjugate

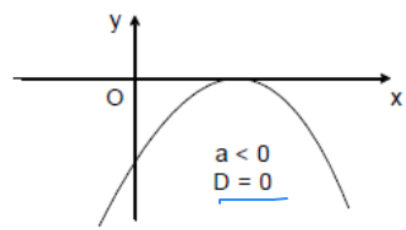
$\left(-\infty, \frac{-D}{4a}\right]$

Fig. 4



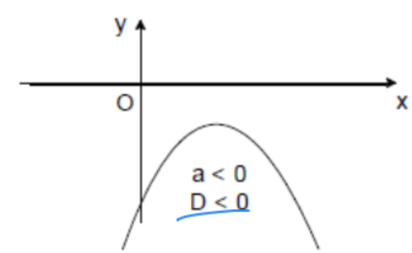
Roots are real & distinct

Fig. 5



Roots are coincident

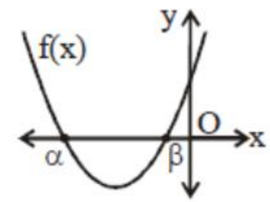
Fig. 6



Roots are complex conjugate

Problems

The following figure shows the graph of $f(x) = ax^2 - bx + c$.
Then which one of the following is correct?



~~(A)~~ $\frac{b}{c} > 0$

~~(B)~~ a and c are of opposite sign

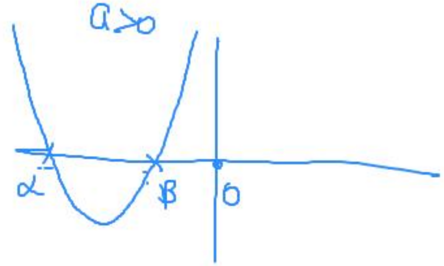
~~(C)~~ a and b are of same sign

(D) None

$f(x) = ax^2 - bx + c$ → α, → β

$\alpha + \beta = \frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

$a > 0$



α & β are $-$ve. Hence $\alpha + \beta = -$ve

$\alpha\beta = +$ve.

$\boxed{\frac{b}{a} < 0 \quad \& \quad \frac{c}{a} > 0}$

$a > 0$, $b < 0$, $c > 0$

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Problems

For all 'x', $x^2 + 2ax + (10 - 3a) > 0$, then the interval in which 'a' lies is

(a) $a < -5$ (b) $-5 < a < 2$ (c) $a > 5$ (d) $2 < a < 5$

$x^2 + 2ax + (10 - 3a) > 0$

leading coeff $= 1 > 0$

$D < 0$

$4a^2 - 4(10 - 3a) < 0$

$a^2 - 10 + 3a < 0$

$a^2 + 3a - 10 < 0$

$(a+5)(a-2) < 0$

$+ \quad - \quad +$

$-5 \quad 2$

MAXIMUM & MINIMUM VALUE of $y = ax^2 + bx + c$ occurs at $x = -(b/2a)$ according as ;

$a < 0$ or $a > 0$. $y \in \left[\frac{4ac - b^2}{4a}, \infty\right)$ if $a > 0$ & $y \in \left(-\infty, \frac{4ac - b^2}{4a}\right]$ if $a < 0$.

$\left[-\frac{D}{4a}, \infty\right)$ $\left(-\infty, -\frac{D}{4a}\right]$

Problems

The value of λ such that sum of the squares of the roots of the quadratic equation, $x^2 + (3 - \lambda)x + 2 = \lambda$ has the least value is **(2019 Main, 10 Jan II)**

(a) $\frac{4}{9}$ (b) 1 (c) $\frac{15}{8}$ (d) 2

$\alpha^2+\beta^2 \longrightarrow$ least

$\alpha+\beta = \lambda-3$

$\alpha\beta = 2-\lambda$

$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (\lambda-3)^2 - 2(2-\lambda)$

$= \lambda^2+9-6\lambda-4+2\lambda$

$= \lambda^2-4\lambda+5$

least value $\rightarrow \frac{-D}{4a}$. at $\lambda = -\frac{b}{2a} = \frac{4}{2} = 2$.

COMMON ROOTS OF 2 QUADRATIC EQUATIONS [ONLY ONE COMMON ROOT] :

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$. Therefore

$a\alpha^2 + b\alpha + c = 0$; $a'\alpha^2 + b'\alpha + c' = 0$. By Cramer's Rule $\boxed{\dfrac{\alpha^2}{bc' - b'c} = \dfrac{\alpha}{a'c - ac'} = \dfrac{1}{ab' - a'b}}$

Therefore, $\alpha = \dfrac{ca' - c'a}{ab' - a'b} = \dfrac{bc' - b'c}{a'c - ac'}$.

$\alpha^2 = \dfrac{bc' - b'c}{ab' - a'b}$, $\alpha = \dfrac{a'c - ac'}{ab' - a'b}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

$ax^2 + bx + c = 0$ α, β

$a'x^2 + b'x + c' = 0$ α, γ

$a\alpha^2 + b\alpha + c = 0$

$a'\alpha^2 + b'\alpha + c' = 0$

Condition for both the common roots :

If both roots of the given equations are common then $\boxed{\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}}$.

$a_1x^2 + b_1x + c_1 = 0$ α, β

$a_2x^2 + b_2x + c_2 = 0$ α, β

Problems

If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$ have a common root, then $a : b : c$ is

both roots common

(a) 1 : 2 : 3 (b) 3 : 2 : 1 **(2013 Main)**

(c) 1 : 3 : 2 (d) 3 : 1 : 2

$x^2+2x+3=0 \rightarrow D = 4-12 = -ve$ $D<0$ $p+iq,\ p-iq$

$ax^2+bx+c=0$ $p+iq,\ p-iq$

$\frac{a}{1} = \frac{b}{2} = \frac{c}{3}$

$a:b:c = 1:2:3$

THEORY OF EQUATIONS : (Relation b/w roots & coefficients of a polynomial eqn)

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots\alpha_n$ are the roots of the equation;

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ where $a_0, a_1, \ldots a_n$ are all real & $a_0 \neq 0$ then,

$$\Sigma \alpha_1 = -\frac{a_1}{a_0},\ \Sigma \alpha_1 \alpha_2 = +\frac{a_2}{a_0},\ \Sigma \alpha_1 \alpha_2 \alpha_3 = -\frac{a_3}{a_0},\ \ldots\ldots,\ \alpha_1 \alpha_2 \alpha_3 \ldots\ldots\alpha_n = (-1)^n \frac{a_n}{a_0}$$

$ax^2+bx+c=0 \rightarrow \alpha, \beta$

$\alpha+\beta = -b/a$, $\alpha\beta = c/a$

$ax^3+bx^2+cx+d=0 \rightarrow \alpha, \beta, \gamma$

$\alpha+\beta+\gamma = -b/a$

$\alpha\beta+\beta\gamma+\gamma\alpha = c/a$

$\alpha\beta\gamma = -d/a$

$ax^4+bx^3+cx^2+dx+e=0$

$\alpha+\beta+\gamma+\omega = -b/a$

$\alpha\beta+\alpha\gamma+\alpha\omega+\beta\gamma+\beta\omega+\gamma\omega = c/a$

$\alpha\beta\gamma+\alpha\beta\omega+\alpha\gamma\omega+\beta\gamma\omega = -d/a$

$\alpha\beta\gamma\omega = e/a$

If a, b, c are the roots of cubic $x^3 - x^2 + 1 = 0$ then find the value of $a^{-2} + b^{-2} + c^{-2}$.

$a+b+c = 1$

$ab+bc+ca = 0$

$abc = -1$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{b^2c^2+a^2c^2+a^2b^2}{a^2b^2c^2} = \frac{(ab)^2+(bc)^2+(ca)^2}{(-1)^2} = \frac{2}{1} = 2$$

$(ab)^2+(bc)^2+(ca)^2 = ?$

$(ab+bc+ca)^2 = (ab)^2+(bc)^2+(ca)^2+2(ab^2c+bc^2a+ca^2b)$

$0 = E + 2abc(a+b+c) \Rightarrow 0 = E + (-2) \Rightarrow E = 2$

$f(x) = x^2 - 3x + 2 = (x-1)(x-2)$

$x = 1 \quad (x-1)$

$f(x) = x^3 + 3x^2 + 3x + 1 = (x+1)(\text{quad.})$

$x = -1$ → one factor $= x+1$

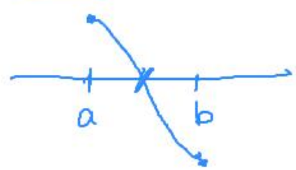
Note : **(i)** If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$ and conversely.

(ii) Every equation of nth degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs.**

(iv) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not a perfect square.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.



$f(x) = x^2 - 3x + 2 = (x-1)(x-2)$ $\qquad$ $f(x) = x^3 + 3x^2 + 3x + 1 = (x+1)(\text{quad.})$

$x = 1$ $\qquad$ $(x-1)$ $\qquad$ $x = -1$ $\qquad$ one factor $= x+1$

Note : (i) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$ and conversely.

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