Properties of Triangles
Heights & Distances

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Properties of Triangle

I. **SINE FORMULA :** In any triangle ABC, $\boxed{\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}}$.

II. **COSINE FORMULA :** (i) $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc.\cos A$

(ii) $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$ (iii) $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$\cos A = \dfrac{b^2+c^2-a^2}{2bc}$, $\cos B = \dfrac{c^2+a^2-b^2}{2ca}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

Problems

$a-d,\ a,\ a+d$

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The angles A, B and C of a ΔABC are in AP and $a : b = 1 : \sqrt{3}$. If $c = 4$ cm, then the area (in sq cm) of this triangle is **(2019 Main, 10 April II)**

(a) $\frac{2}{\sqrt{3}}$ (b) $4\sqrt{3}$ (c) $2\sqrt{3}$ (d) $\frac{4}{\sqrt{3}}$

A, B, C are in A.P

$(B-d,\ B,\ B+d)$ 3 angles

$B-d+B+B+d = 180^\circ$

$3B = 180^\circ$

$B = 60^\circ$

$\frac{a}{b} = \frac{1}{\sqrt{3}}$

$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{a}{b} = \frac{\sin A}{\sin B} \Rightarrow \frac{1}{\sqrt{3}} = \frac{\sin A}{\sin 60^\circ} \Rightarrow \sin A = \frac{\sin 60^\circ}{\sqrt{3}} = \frac{\sqrt{3}/2}{\sqrt{3}} = \frac{1}{2}$

$A = 30^\circ$

$A = 30^\circ,\ B = 60^\circ,\ C = 90^\circ$

Area $= \frac{1}{2} a \times b$

$= \frac{1}{2} \times 2 \times 2\sqrt{3}$

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow \frac{a}{1/2} = \frac{b}{\sqrt{3}/2} = \frac{4}{1}$

$a = 2,\ b = 2\sqrt{3}$

Problems

Given, $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ for a ΔABC with usual notation. If $\frac{\cos A}{\alpha} = \frac{\cos B}{\beta} = \frac{\cos C}{\gamma}$, then the ordered triad (α, β, γ) has a value **(2019 Main, 11 Jan II)**

(a) (19, 7, 25) (b) (3, 4, 5)

(c) (5, 12, 13) (d) (7, 19, 25)

$\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = k$

$b+c = 11k$, $c+a = 12k$, $a+b = 13k$
(1) (2) (3)

$2(a+b+c) = 36k$

$a+b+c = 18k \longrightarrow$ (4)

$\cos A : \cos B : \cos C = \alpha : \beta : \gamma$

(4) − (1) $a = 7k$

(4) − (2) $b = 6k$

(4) − (3) $c = 5k$

$\frac{1}{5} : \frac{19}{35} : \frac{5}{7}$

$7 : 19 : 25$

$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36k^2+25k^2-49k^2}{2\times 30k^2} = \frac{12\cancel{k^2}}{60\cancel{k^2}} = 1/5$

$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{25k^2+49k^2-36k^2}{2\times 35k^2} = \frac{38\cancel{k^2}}{70\cancel{k^2}} = \frac{19}{35}$

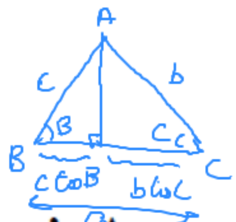
$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49k^2+36k^2-25k^2}{2\times 42k^2} = \frac{60k^2}{84k^2} = \frac{30}{42} = \frac{15}{21} = \frac{5}{7}$

Properties of Triangle

III. PROJECTION FORMULA :

(i) $a = b\cos C + c\cos B$ **(ii)** $b = c\cos A + a\cos C$

(iii) $c = a\cos B + b\cos A$



IV. NAPIER'S ANALOGY – TANGENT RULE :

(i) $\tan\dfrac{B-C}{2} = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}$

(ii) $\tan\dfrac{C-A}{2} = \dfrac{c-a}{c+a}\cot\dfrac{B}{2}$ **(iii)** $\tan\dfrac{A-B}{2} = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

$\boxed{\tan\left(\dfrac{B-C}{2}\right) = \dfrac{b-c}{b+c}\cot\dfrac{A}{2}}$

$\tan\left(\dfrac{A-B}{2}\right) = \dfrac{a-b}{a+b}\cot\dfrac{C}{2}.$

Problems

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With the usual notation, in ΔABC, if $\angle A + \angle B = 120°$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is **(2019 Main, 10 Jan II)**

(a) 7 : 1 (b) 3 : 1
(c) 9 : 7 (d) 5 : 3

$A + B = 120° \Rightarrow C = 60°$

$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot\frac{C}{2}$

$= \frac{\sqrt{3}+1 - (\sqrt{3}-1)}{\sqrt{3}+1 + \sqrt{3}-1} \times \cot\frac{60}{2}$

$= \frac{\cancel{2}}{\cancel{2}\sqrt{3}} \cot 30° = \frac{1}{\sqrt{3}} \times \sqrt{3} = 1$

$\tan\left(\frac{A-B}{2}\right) = 1$

$\frac{A-B}{2} = 45°$

$A - B = 90°$ — (1)

$A + B = 120°$ — (2)

$2A = 210°$

$A = 105°$

$B = 15°$

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Properties of Triangle

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V. Trigonometric Functions Of Half Angles :

(i) $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$; $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$; $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$; $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$; $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)}$ where $s = \frac{a+b+c}{2}$ & $\Delta =$ area of triangle.

(iv) Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$.

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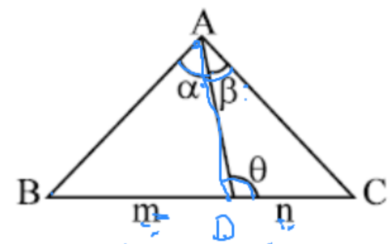
Properties of Triangle

VI. M – N **RULE** : In any triangle,

$$(m+n)\cot\theta = m\cot\alpha - n\cot\beta$$
$$= n\cot B - m\cot C$$



$\Delta = \frac{1}{2} ab \sin C.$

$= \frac{1}{2} ab \frac{c}{2R} = \frac{abc}{4R}$

$\sin C = \frac{c}{2R}$

Area of $\Delta = \frac{1}{2} ab \sin C$

$= \frac{1}{2} ca \sin B$

$= \frac{1}{2} bc \sin A$

VII. $\frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B =$ area of triangle ABC.

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R}$$

radius = R

Note that $R = \frac{abc}{4\Delta}$; Where R is the radius of circumcircle & Δ is area of triangle

$\Delta = \frac{abc}{4R} \Rightarrow \boxed{R = \frac{abc}{4\Delta}}$

Problems

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→ In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is (2019 Main, 11 Jan I)

(a) $\frac{c}{3}$ (b) $\frac{c}{\sqrt{3}}$

(c) $\frac{3}{2}y$ (d) $\frac{y}{\sqrt{3}}$

a, b, c are the sides.

$a+b=x$, $ab=y$

$x^2-c^2=y$

$R = \frac{abc}{4\Delta} = \frac{y \times c}{4 \times \frac{\sqrt{3}y}{4}} = \frac{c}{\sqrt{3}}$

$x^2 - c^2 = y$

$(a+b)^2 - c^2 = ab$.

$a^2+b^2+2ab-c^2 = ab$

$a^2+b^2-c^2 = -ab$

$\boxed{\frac{a^2+b^2-c^2}{2ab}} = \frac{-ab}{2ab} = \frac{-1}{2}$

$\cos C = -\frac{1}{2}$.

$C = 120^\circ$

$\Delta = \frac{1}{2}ab\sin C = \frac{1}{2}y \times \sin 120^\circ = \frac{\sqrt{3}y}{4}$.

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Properties of Triangle

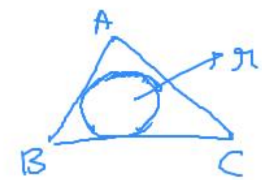
VIII. Radius of the incircle 'r' is given by:

(a) $r = \frac{\Delta}{s}$ where $\boxed{s = \frac{a+b+c}{2}}$

(b) $r = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2} = (s-c)\tan\frac{C}{2}$

(c) $r = \frac{a \sin\frac{B}{2} \sin\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(d) $r = 4R \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$



IX. Radius of the Ex–circles r_1, r_2 & r_3 are given by :

(a) $r_1 = \frac{\Delta}{s-a}$; $r_2 = \frac{\Delta}{s-b}$; $r_3 = \frac{\Delta}{s-c}$

(b) $r_1 = s\tan\frac{A}{2}$; $r_2 = s\tan\frac{B}{2}$; $r_3 = s\tan\frac{C}{2}$

(c) $r_1 = \frac{a \cos\frac{B}{2} \cos\frac{C}{2}}{\cos\frac{A}{2}}$ & so on

(d) $r_1 = 4R\sin\frac{A}{2} \cdot \cos\frac{B}{2} \cdot \cos\frac{C}{2}$;

$r_2 = 4R\sin\frac{B}{2} \cdot \cos\frac{A}{2} \cdot \cos\frac{C}{2}$;

$r_3 = 4R\sin\frac{C}{2} \cdot \cos\frac{A}{2} \cdot \cos\frac{B}{2}$

Problems

In a ΔABC, let $\angle C = \pi / 2$. If r is the inradius and R is the circumradius of the triangle, then $2(r + R)$ is equal to

(a) $a + b$ (b) $b + c$ (c) $c + a$ (d) $a + b + c$

$r = (s-c) \tan \frac{C}{2}$

$= \left(\frac{a+b+c}{2} - c\right) \tan \frac{\pi}{4}$

$= \frac{a+b+c-2c}{2}$

$r = \frac{a+b-c}{2}$

$\boxed{2r = a+b-c}$ — (1)

$\frac{a}{\sin A} = \frac{b}{\sin B} = \boxed{\frac{c}{\sin C} = 2R}$

$\frac{c}{\sin \frac{\pi}{2}} = 2R$

$\boxed{2R = c}$ — (2)

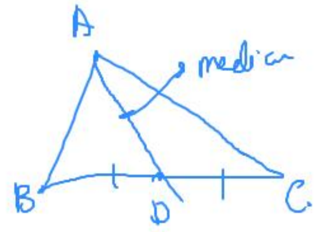
$2r + 2R = a + b - \cancel{c} + \cancel{c}$

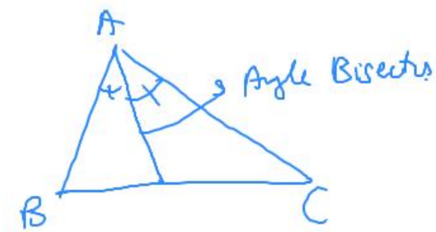
X. LENGTH OF ANGLE BISECTOR & MEDIANS :

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$$\boxed{m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}} \text{ and } \boxed{\beta_a = \frac{2bc\cos\frac{A}{2}}{b + c}}$$

Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$





Problems

The sides of a triangle are in the ratio $1:\sqrt{3}:2$, then the angles of the triangle are in the ratio

(a) 1 : 3 : 5 (b) 2 : 3 : 2 (c) 3 : 2 : 1 (d) 1 : 2 : 3

$a:b:c = \sin A : \sin B : \sin C$

$1:\sqrt{3}:2 = \sin A : \sin B : \sin C$

$\frac{1}{2}:\frac{\sqrt{3}}{2}:1 = \sin A : \sin B : \sin C$

$A = 30^\circ, B = 60^\circ, C = 90^\circ$

$30^\circ : 60^\circ : 90^\circ$

$1:2:3$

Heights & Distances

SOME TERMINOLOGY RELATED TO HEIGHT AND DISTANCE

Angle of elevation and angle of depression

If a horizontal line is drawn from the eye of the observer (O) and an object P is above this line OX, then $\angle POX$ is called **angle of elevation.**

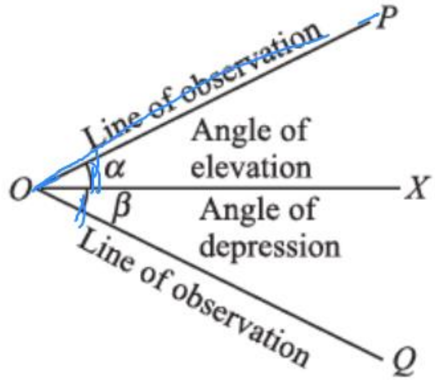


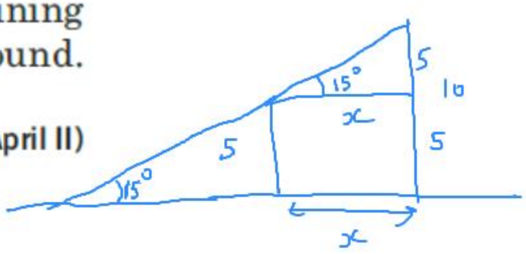
Fig. 28.1

If an object Q is below the horizontal line OX, then $\angle QOX$ is called **angle of depression.**

Problems

Two poles standing on a horizontal ground are of heights 5 m and 10 m, respectively. The line joining their tops makes an angle of 15° with the ground. Then, the distance (in m) between the poles, is

(2019 Main, 9 April II)



(a) $5(\sqrt{3}+1)$ (b) $\frac{5}{2}(2+\sqrt{3})$

(c) $10(\sqrt{3}-1)$ (d) $5(2+\sqrt{3})$

$\tan 15^\circ = \frac{5}{x}$ $\Rightarrow x = \frac{5}{\tan 15^\circ}$ $= 5 \cot 15^\circ$

$= 5(2+\sqrt{3})$

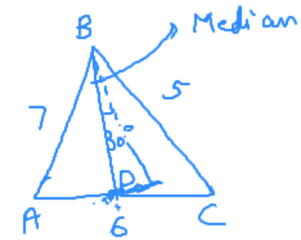
$\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$

$\cot 15^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

Problems

Consider a triangular plot ABC with sides $AB = 7$ m, $BC = 5$ m and $CA = 6$ m. A vertical lamp-post at the mid-point D of AC subtends an angle 30° at B. The height (in m) of the lamp-post is **(2019 Main, 10 Jan I)**

(a) $\frac{2}{3}\sqrt{21}$ (b) $2\sqrt{21}$ (c) $7\sqrt{3}$ (d) $\frac{3}{2}\sqrt{21}$



length of Median (from B) = $\frac{1}{2}\sqrt{2c^2+2a^2-b^2}$

$b = 6$, $a = 5$, $c = 7$

$BD = \frac{1}{2}\sqrt{98+50-36} = \frac{1}{2}\sqrt{112}$

$= \frac{2}{2}\sqrt{28} = 2\sqrt{7}$

$h = \frac{BD}{\sqrt{3}} = \frac{2\sqrt{7}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2}{3}\sqrt{21}$

$\tan 30^\circ = \frac{h}{BD}$

$h = BD \tan 30^\circ$

$= BD \times \frac{1}{\sqrt{3}}$

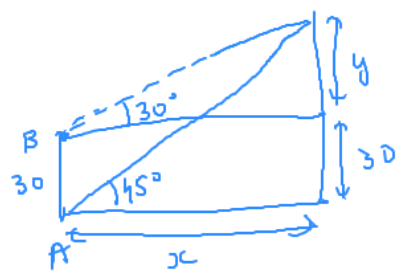
Problems

The angle of elevation of the top of a vertical tower standing on a horizontal plane is observed to be 45° from a point A on the plane. Let B be the point 30 m vertically above the point A. If the angle of elevation of the top of the tower from B be 30°, then the distance (in m) of the foot of the tower from the point A is

(2019 Main, 12 April II)

(a) $15(3+\sqrt{3})$ (b) $15(5-\sqrt{3})$

(c) $15(3-\sqrt{3})$ (d) $15(1+\sqrt{3})$



$\tan 45^\circ = \frac{30+y}{x}$

$\boxed{30+y=x}$

$\tan 30^\circ = \frac{y}{x} \Rightarrow y = \frac{x}{\sqrt{3}}$

$30+y=x$

$30+\frac{x}{\sqrt{3}}=x$

$30 = x\left(1-\frac{1}{\sqrt{3}}\right)$

$30 = \frac{x(\sqrt{3}-1)}{\sqrt{3}}$

$x = \frac{30\sqrt{3}}{(\sqrt{3}-1)}\frac{(\sqrt{3}+1)}{(\sqrt{3}+1)} = \frac{30(3+\sqrt{3})}{2}$

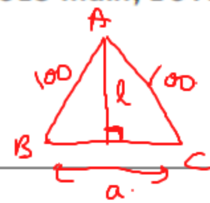
$= 15(3+\sqrt{3})$

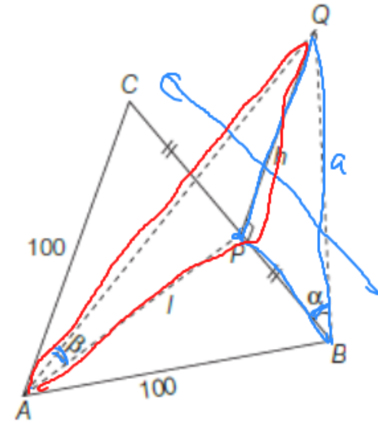
Problems

ABC is a triangular park with $AB = AC = 100$ m. A vertical tower is situated at the mid-point of BC. If the angles of elevation of the top of the tower at A and B are $\cot^{-1}(3\sqrt{2})$ and $\operatorname{cosec}^{-1}(2\sqrt{2})$ respectively, then the height of the tower (in m) is **(2019 Main, 10 April I)**

(a) 25 (b) 20

(c) $10\sqrt{5}$ (d) $\dfrac{100}{3\sqrt{3}}$





$\tan\beta = \dfrac{h}{l}$

$h = l\tan\beta$

$\boxed{h = \dfrac{l}{3\sqrt{2}}}$

$l = 3\sqrt{2}h$

$l^2 = 18h^2$

$\cot^{-1} 3\sqrt{2} = \beta$

$\cot\beta = 3\sqrt{2}$

$\tan\beta = \dfrac{1}{3\sqrt{2}}$

$\operatorname{cosec}^{-1} 2\sqrt{2} = \alpha$

$\operatorname{cosec}\alpha = \dfrac{2\sqrt{2}}{1}$

$\sin\alpha = \dfrac{1}{2\sqrt{2}} = \dfrac{p}{h}$

$b = \sqrt{8-1} = \sqrt{7}$

$\tan\alpha = \dfrac{1}{\sqrt{7}}$

$\Delta\, QPB$

$\tan\alpha = \dfrac{h}{PB}$

$\dfrac{1}{\sqrt{7}} = \dfrac{h}{PB} \Rightarrow \boxed{h = \dfrac{PB}{\sqrt{7}} = \dfrac{a/2}{\sqrt{7}} = \dfrac{a}{2\sqrt{7}}} \Rightarrow a = 2\sqrt{7}h$

$a^2 = 28h^2$

$l^2 + \left(\dfrac{a}{2}\right)^2 = 100^2$

$l^2 + \dfrac{a^2}{4} = 100^2 \Rightarrow 18h^2 + 7h^2 = 100^2$

$\Rightarrow 25h^2 = 100\times 100 \Rightarrow h^2 = 4\times 100 = 400$

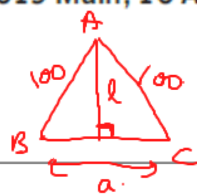
$h = 20$

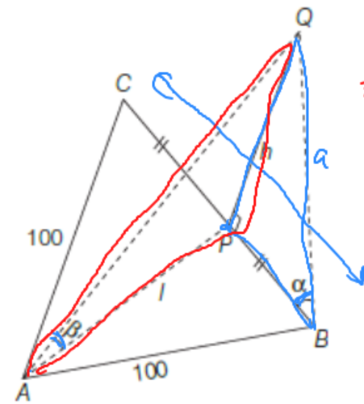
Problems

ABC is a triangular park with $AB = AC = 100$ m. A vertical tower is situated at the mid-point of BC. If the angles of elevation of the top of the tower at A and B are $\cot^{-1}(3\sqrt{2})$ and $\operatorname{cosec}^{-1}(2\sqrt{2})$ respectively, then the height of the tower (in m) is **(2019 Main, 10 April I)**

(a) 25 (b) 20

(c) $10\sqrt{5}$ (d) $\dfrac{100}{3\sqrt{3}}$





$\tan B = \dfrac{h}{l}$

$h = l \tan B$

$\boxed{h = \dfrac{l}{3\sqrt{2}}}$

$l = 3\sqrt{2}h$

$l^2 = 18h^2$

$\cot^{-1} 3\sqrt{2} = B$

$\cot B = 3\sqrt{2}$

$\tan B = \dfrac{1}{3\sqrt{2}}$

$\operatorname{cosec}^{-1} 2\sqrt{2} = \alpha$

$\operatorname{cosec}\alpha = \dfrac{2\sqrt{2}}{1}$

$\sin\alpha = \dfrac{1}{2\sqrt{2}} = \dfrac{p}{h}$

$b = \sqrt{8-1} = \sqrt{7}$

$\tan\alpha = \dfrac{1}{\sqrt{7}}$

$\Delta\ QPB$

$\tan\alpha = \dfrac{h}{PB}$

$\dfrac{1}{\sqrt{7}} = \dfrac{h}{PB} \Rightarrow \boxed{h = \dfrac{PB}{\sqrt{7}} = \dfrac{a/2}{\sqrt{7}} = \dfrac{a}{2\sqrt{7}}} \Rightarrow a = 2\sqrt{7}h$

$a^2 = 28h^2$

$l^2 + \left(\dfrac{a}{2}\right)^2 = 100^2$

$l^2 + \dfrac{a^2}{4} = 100^2 \Rightarrow 18h^2 + 7h^2 = 100^2$

$\Rightarrow 25h^2 = 100\times100 \Rightarrow h^2 = 4\times100 = 400$

$h = 20$