# Trigonometric Equations

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An equation (or inequality) involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation.

$$\cos x = \frac{1}{2}; \sin^2 x - 4\cos x = 1.$$

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The value of an unknown angle which satisfies the given trigonometric equation is called a solution or root of the equation.

$$2\sin\theta = \sqrt{3}$$

How many solutions ?

$\sin\theta = \sqrt{3}/2 \quad \theta = 60°, 120°$

Clearly = 60° and 120° are solutions of the equation between 0° and 360°.

How many solutions overall ???

There are millions and billions of solutions which satisfy say $\tan x = -1$ but our main object is to write down those infinite solution in one line. Since all trigonometric function are periodic and therefore solution of all trigonometrical equation can be generalised with the help of periodicity of trigonometrical function.

**Two types of solution**

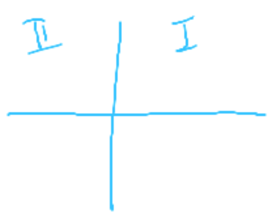
- Principal solution
  $0 \le \theta < 2\pi$
- General solution
  All solns

# Principal Solution

The solutions of a trigonometric equation lying in the interval $[0, 2\pi)$. For example, $\sin\theta = \frac{1}{2}$, then the two values of $\theta$ between 0 and $2\pi$ are $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. Thus, $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are the principal solutions of equation $\sin\theta = \frac{1}{2}$.

$\theta = 30^\circ, 180^\circ - 30^\circ$

$30^\circ, 150^\circ$



# General Solution

The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

# Trigonometric Equations

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1. If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$ .
2. If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$ .
3. If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$ .
4. If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
5. $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.
6. $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

[ Note : $\alpha$ is called the principal angle ]

## TYPES OF TRIGONOMETRIC EQUATIONS :

(a) Solutions of equations by factorising. Consider the equation ;
$(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$ ; $\cot x - \cos x = 1 - \cot x \cos x$

(b) Solutions of equations reducible to quadratic equations. Consider the equation :
$3\cos^2 x - 10\cos x + 3 = 0$ and $2\sin^2 x + \sqrt{3}\sin x + 1 = 0$

(c) Solving equations by introducing an Auxilliary argument. Consider the equation :
$\sin x + \cos x = \sqrt{2}$ ; $\sqrt{3}\cos x + \sin x = 2$ ; $\sec x - 1 = (\sqrt{2} - 1)\tan x$


* a) $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$(2\sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$(1 + \cos x)[2\sin x - \cos x - (1 - \cos x)] = 0$

$(1 + \cos x)(2\sin x - \cancel{\cos x} - 1 + \cancel{\cos x}) = 0$

$(1 + \cos x)(2\sin x - 1) = 0$

$\cos x = -1$ , $\sin x = \frac{1}{2}$

b) $3\cos^2 x - 10\cos x + 3 = 0$

$3\cos^2 x - 9\cos x - \cos x + 3 = 0$

$(3\cos x - 1)(\cos x - 3) = 0$

$\boxed{\cos x = \frac{1}{3}}$ , $\cos x = 3$ ✗

$\cos x = \cos\left(\cos^{-1}\frac{1}{3}\right) \Rightarrow \boxed{x = 2n\pi \pm \cos^{-1}\frac{1}{3}}$

c) $a\cos x + b\sin x = c$

$\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos x + \frac{b}{\sqrt{a^2+b^2}}\sin x\right)$

$\sqrt{a^2+b^2}(\sin\alpha\cos x + \cos\alpha\sin x)$

$\sqrt{a^2+b^2}(\sin(x+\alpha)) = c$

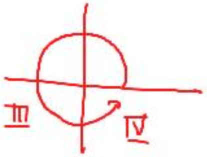
# Problems

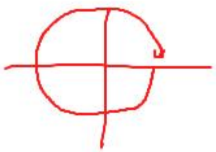
Let $S = \{\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0\}$, then sum of the elements of $S$ is **(2019 Main)**

(a) $2\pi$ (b) $\pi$

(c) $\dfrac{5\pi}{3}$ (d) $\dfrac{13\pi}{6}$

$-2\pi \le \theta \le 2\pi$



$2\cos^2\theta + 3\sin\theta = 0$

$2(1 - \sin^2\theta) + 3\sin\theta = 0$

$2 - 2\sin^2\theta + 3\sin\theta = 0$

$2\sin^2\theta - 3\sin\theta - 2 = 0$

$2\sin^2\theta - 4\sin\theta + \sin\theta - 2 = 0$

$2\sin\theta(\sin\theta - 2) + 1(\sin\theta - 2) = 0$

$(2\sin\theta + 1)(\sin\theta - 2) = 0$ $\quad \sin\theta = 2, -\frac{1}{2}$

$\boxed{\sin\theta = -\frac{1}{2}}$ $\rightarrow$ $\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$

$\theta = 180 + 30^\circ,\ 360 - 30^\circ$

$= 210^\circ, 330^\circ$

$\theta = -30^\circ, -(150^\circ)$

$210^\circ + 330^\circ - 30^\circ - 150^\circ = 360^\circ$

$n = 0$ $\boxed{-\frac{\pi}{6}}$

$n = 1$ $\boxed{\pi + \frac{\pi}{6}}$

$n = 2$ $\boxed{2\pi - \frac{\pi}{6}}$

$n = 3$ $3\pi + \frac{\pi}{6}$

$n = -1$ $\boxed{-\pi + \frac{\pi}{6}}$

# Problems

The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is (2019 Main, 10 Jan I)

(a) $\frac{3\pi}{8}$ (b) $\frac{5\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\pi$

$\cos^2\theta = \cos^2\alpha$

$\theta = n\pi \pm \alpha$

$\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$

$1 - \cos^2 2\theta + \cos^4 2\theta = \frac{3}{4}$

$4 - 4\cos^2 2\theta + 4\cos^4 2\theta = 3$

$4\cos^4 2\theta - 4\cos^2 2\theta + 1 = 0$

$(2\cos^2 2\theta - 1)^2 = 0$

$2\cos^2 2\theta = 1$

$\cos^2 2\theta = \frac{1}{2}$

$\cos^2 2\theta = \left(\frac{1}{\sqrt{2}}\right)^2 = \cos^2\frac{\pi}{4}$

$2\theta = n\pi \pm \frac{\pi}{4}$

$\boxed{\theta = \frac{n\pi}{2} \pm \frac{\pi}{8}}$

$n = -1 \quad -\frac{\pi}{2} \pm \frac{\pi}{8}$ ✗

$\boxed{n = 0 \quad \frac{\pi}{8}}$ ✓

$\boxed{n = 1 \quad \frac{\pi}{2} - \frac{\pi}{8}}$ ✓

$n = 2 \quad \pi \pm \frac{\pi}{8}$ ✗

Ans: $\frac{\pi}{8}, \frac{\pi}{2} - \frac{\pi}{8}$

Sum $\pi/2$

## TYPES OF TRIGONOMETRIC EQUATIONS :

(d) Solving equations by Transforming a sum of Trigonometric functions into a product.
Consider the example : $\cos 3x + \sin 2x - \sin 4x = 0$ ;
$\sin^2 x + \sin^2 2x + \sin^2 3x + \sin^2 4x = 2$ ; $\sin x + \sin 5x = \sin 2x + \sin 4x$

(e) Solving equations by transforming a product of trigonometric functions into a sum.
Consider the equation :

$$\sin 5x \, . \cos 3x = \sin 6x \, .\cos 2x \; ; \; 8 \cos x \cos 2x \cos 4x = \frac{\sin 6x}{\sin x} \; ; \; \sin 3\theta = 4 \sin\theta \, \sin 2\theta \, \sin 4\theta$$

(f) Solving equations with the use of the Boundness of the functions

d) $\cos 3x + \underbrace{\sin 2x - \sin 4x}_{\sin C - \sin D} = 0$

e) $2\sin 5x \cdot \cos 3x = 2\sin 6x \cdot \cos 2x$

$2\sin A \cos B$

$\vdots$

** $\boxed{\underbrace{\sin x}_{\le 1} = \underbrace{1+y^2}_{\ge 1}}$

$\sin x = 1$
& $1+y^2 = 1$

Boundness of function.

$-1 \le \sin x \le 1$

$-1 \le \cos x \le 1$

$\sin x \ge 1$

$\Rightarrow \sin x = 1$

# Problems

If $0 \le x < \frac{\pi}{2}$, then the number of values of $x$ for which $\sin x - \sin 2x + \sin 3x = 0$, is (2019 Main, 9 Jan II)

(a) 2 (b) 3 (c) 1 (d) 4

$\sin x - \sin 2x + \sin 3x = 0$

$2\sin 2x \cdot \cos x - \sin 2x = 0$

$\sin 2x (2\cos x - 1) = 0$

$\sin 2x = 0 \qquad \cos x = \frac{1}{2}$

$2x = n\pi. \qquad \boxed{x = \frac{\pi}{3}}$

$x = \frac{n\pi}{2}$

$\boxed{x=0}$

# Problems



The number of solutions of the equation

$1 + \sin^4 x = \cos^2 3x, x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is **(2019 Main, 12 April I)**

(a) 3 (b) 5

(c) 7 (d) 4

$\underbrace{1 + \sin^4 x}_{\geq 1} = \underbrace{\cos^2 3x}_{\leq 1}$

$1 + \sin^4 x \geq 1$

$\cos^2 3x \leq 1$

Solution will exist when

Both LHS & RHS should be equal to 1

$-1 \leq \cos x \leq 1$

$0 \leq \cos^2 x \leq 1$

$1 + \sin^4 x = 1$ & $\boxed{\cos^2 3x = 1}$ $\Rightarrow$ $1 - \sin^2 3x = 1$

$\boxed{\sin^4 x = 0}$ ∴

$\sin^2 3x = 0$

$\boxed{\sin 3x = 0}$

$\sin x = 0$

$\underbrace{x = 0, \pi, 2\pi, -\pi, -2\pi}$

# Problems

Let $\alpha$ and $\beta$ be the roots of the quadratic equation $x^2 \sin\theta - x(\sin\theta\cos\theta + 1) + \cos\theta = 0$ $(0 < \theta < 45^\circ)$ and $\alpha < \beta$. Then, $\sum_{n=0}^{\infty}\left(\alpha^n + \frac{(-1)^n}{\beta^n}\right)$ is equal to

**(2019 Main, 11 Jan II)**

(a) $\frac{1}{1-\cos\theta} - \frac{1}{1+\sin\theta}$ (b) $\frac{1}{1-\cos\theta} + \frac{1}{1+\sin\theta}$

(c) $\frac{1}{1+\cos\theta} - \frac{1}{1-\sin\theta}$ (d) $\frac{1}{1+\cos\theta} + \frac{1}{1-\sin\theta}$

$\alpha^0 + \frac{1}{\beta^0} + \alpha^1 - \frac{1}{\beta^1} + \alpha^2 + \frac{1}{\beta^2} + \alpha^3 - \frac{1}{\beta^3} \text{ --- } \infty$

$(\alpha^0 + \alpha^1 + \alpha^2 \text{ --- } \infty) + \left(\frac{1}{\beta^0} - \frac{1}{\beta^1} + \frac{1}{\beta^2} \text{ --- } \infty\right)$

$\frac{1}{1-\alpha} + \frac{1}{1-\left(-\frac{1}{\beta}\right)} = \frac{1}{1-\alpha} + \frac{1}{1+\frac{1}{\beta}} = \frac{1}{1-\cos\theta} + \frac{1}{1+\sin\theta}$

$x^2\sin\theta - x\sin\theta\cos\theta - x + \cos\theta = 0$

$x\sin\theta(x-\cos\theta) - 1(x-\cos\theta) = 0$

$(x\sin\theta - 1)(x-\cos\theta) = 0$

$x\sin\theta = 1$ $\quad x = \cos\theta$

$x = \frac{1}{\sin\theta}$ $\quad x = \cos\theta$

$x = \operatorname{cosec}\theta$, $x = \cos\theta$

$\boxed{\alpha = \cos\theta, \ \beta = \operatorname{cosec}\theta}$

$\frac{1}{\beta} = \sin\theta$

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**Note : Some standard identities in triangle ($A + B + C = \pi$) :** Conditional Identity.

$A+B+C=\pi$

(1) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(2) $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$

(3) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$

(4) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$

(5) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(6) $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{C}{2}\tan\frac{B}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$

Sum of sines or cosines of n angles,

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots\ldots+\sin\left(\alpha+\overline{n-1}\,\beta\right)=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha+\frac{n-1}{2}\beta\right)$$

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots\ldots+\cos\left(\alpha+\overline{n-1}\,\beta\right)=\frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha+\frac{n-1}{2}\beta\right)$$

Angles are in A.P

$\sin 10^\circ + \sin 30^\circ + \sin 50^\circ + \sin 70^\circ + \sin 90^\circ + \sin 110^\circ + \sin 130^\circ$

$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) --- \sin(\alpha+(n-1)\beta) = \dfrac{\sin n\frac{\beta}{2}}{\sin\frac{\beta}{2}} \cdot \sin\left(\alpha+(n-1)\frac{\beta}{2}\right)$

First term of A.P $= \alpha$

c.d of A.P $= \beta$

no. of terms $= n$

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) --- \cos(\alpha+(n-1)\beta) = \dfrac{\sin n\frac{\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha+(n-1)\frac{\beta}{2}\right)$

# Problems

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*Find the sum of series* $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$.

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) - \text{---}$

$\frac{10\pi}{11} + \frac{\pi}{11} = \frac{11\pi}{11} = \pi.$

Angles are in A.P

$\alpha = \frac{\pi}{11}$, $\beta = \frac{2\pi}{11}$  $n=5$

$S = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cdot \cos\left(\alpha + (n-1)\frac{\beta}{2}\right)$

$\text{Sum} = \frac{\sin\frac{5\pi}{11}}{\sin\frac{\pi}{11}} \cdot \cos\left(\frac{\pi}{11} + 4\times\frac{\pi}{11}\right) = \frac{2\sin\frac{5\pi}{11}\cdot\cos\frac{5\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{\sin\frac{10\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{\sin\left(\pi-\frac{\pi}{11}\right)}{2\sin\frac{\pi}{11}}$

$= \frac{\cancel{\sin\pi/11}}{2\cancel{\sin\pi/11}} = \frac{1}{2}.$

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