Complex Numbers

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Complex Numbers: Properties of *i* (iota)

1. $i = \sqrt{-1}$
2. $i^2 = -1$
3. $i^3 = -i$
4. $i^4 = 1$

1. $i^{4n+1} = i \quad n \in I$
2. $i^{4n+2} = i^2 = -1 \quad n \in I$
3. $i^{4n+3} = i^3 = -i \quad n \in I$
4. $i^{4n} = 1 \quad n \in I$
5. $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0 \quad n \in I$

NOTE: All Real numbers are complex numbers with imaginary part as zero

NOTE: Inequality of complex numbers is not defined i.e. $3 + 4i > 1 + 2i$ does not make any sense.

Complex Numbers: Algebra Of Complex Number

1. Addition: $(a+ib)+(c+id)=(a+c)+i(b+d)$

2. Subtraction: $(a+ib)-(c+id)=(a-c)+i(b-d)$

3. Multiplication: $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$

4. Division : $\frac{c+id}{a+ib}=\frac{(c+id)(a-ib)}{a^2+b^2}$

5. Reciprocal : $\frac{1}{a+ib}=\frac{a-ib}{a^2+b^2}$

NOTE: If a complex number is in reciprocal, always rationalize the denominator to get it in form of $a \pm ib$

NOTE:- $a+ib=c+id \Rightarrow a=c \text{ and } b=d$

Complex Numbers: Conjugate Of Complex number

$z = a + ib$ then $\bar{Z}$ is conjugate of z and $\bar{Z} = a - ib$

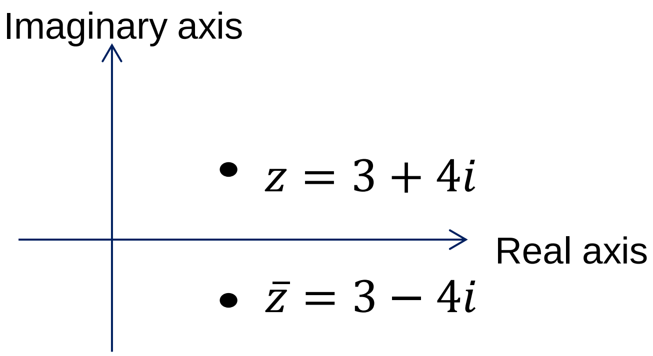
To find conjugate of a complex number, change sign of imaginary part of z

Eg. $\overline{3+4i} = 3 - 4i$

$\overline{-3-4i} = -3 + 4i$

$\overline{3-4i} = 3 + 4i$

$\overline{-3+4i} = -3 - 4i$



Note: Conjugate of a complex number is mirror image of it in real axis

Complex Numbers: Properties Of Conjugate

- $\bar{\bar{z}} = z$
- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
- $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$
- $\overline{(z)^n} = (\bar{z})^n$
- $z_1\bar{z}_2 + z_2\bar{z}_1 = 2Re(\bar{z}_1 z_2) = 2Re(z_1\bar{z}_2)$

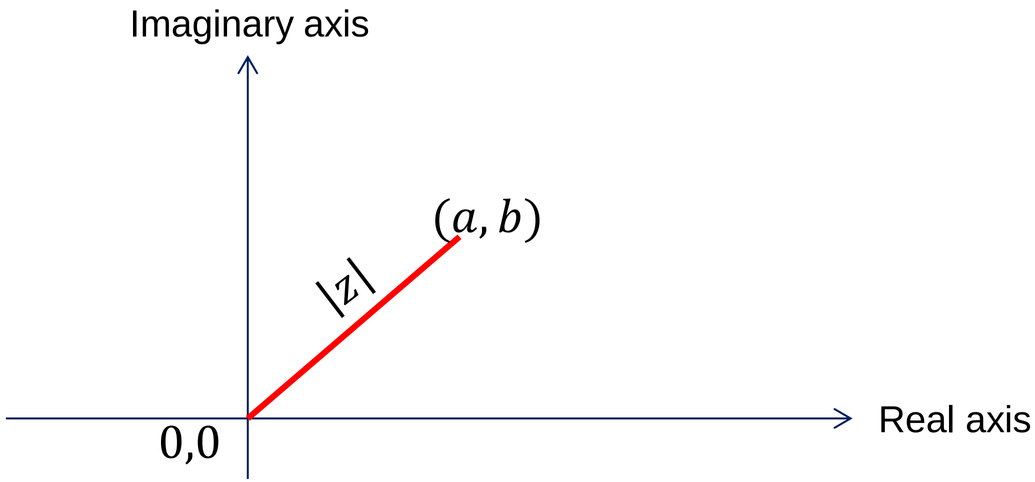
$Im(z_1 z_2) = 0 \Rightarrow z_1 \; and \; z_2 \; are \; conjugate \; of \; each \; other$

Complex Numbers: Modulus Of Complex Number

Modulus of a complex number $z = a + ib$ is **distance of it from origin**

i.e. $|z| = \sqrt{a^2 + b^2}$

Modulus is also referred as absolute value.



Complex Numbers: Properties Of Modulus 1

- $|z| \geq 0 \Rightarrow |z| = 0\ and |z| > 0\ if\ z \neq 0$
- $-|z| \leq Re(z) \leq |z|$ and $-|z| \leq im(z) \leq |z|$
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z.\bar{z} = |z|^2$
- $|z_1 z_2 z_3 \ldots\ldots zn| = |z_1||z_2| \ldots \ldots\ldots |z_n|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $|z^n| = |z|^n$

Complex Numbers: Properties Of Modulus 2

- $|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$

  $= (z_1 + z_2)\ (\bar{z}_1 + \bar{z}_2)$

  $= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$

  $= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$

- $|z_1 - z_2|^2 = (z_1 - z_2)(\overline{z_1 - z_2})$

  $= (z_1 - z_2).(\bar{z}_1 - \bar{z}_2)$

  $= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - z_2\bar{z}_1$

  $= |z_1|^2 + |z_2|^2 - 2Re(z_1\bar{z}_2)$

- $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

- $z_1\bar{z}_2 + z_2\bar{z}_1 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

Complex Numbers: Modulus Inequality

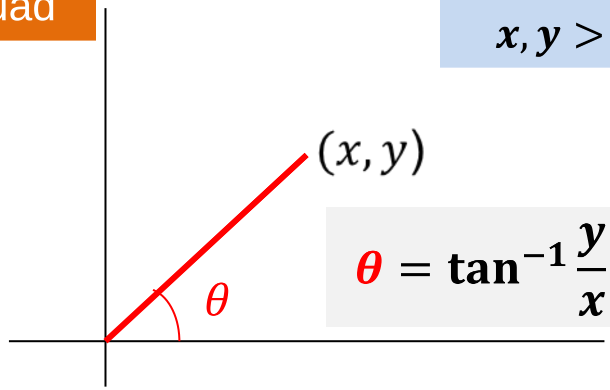
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $|z_1 - z_2| \leq |z_1| + |z_2|$
- $|z_1 + z_2| \geq \big||z_1| - |z_2|\big|$
- $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$
- $|z_1 + z_2 + .\ldots\ldots. + z_n| \leq |z_1| + |z_2| + ..\ldots + |z_n|$

Complex Numbers: Principal Argument or Amplitude

$$-\pi < \theta \leq \pi \qquad z = x + iy$$

1st quad

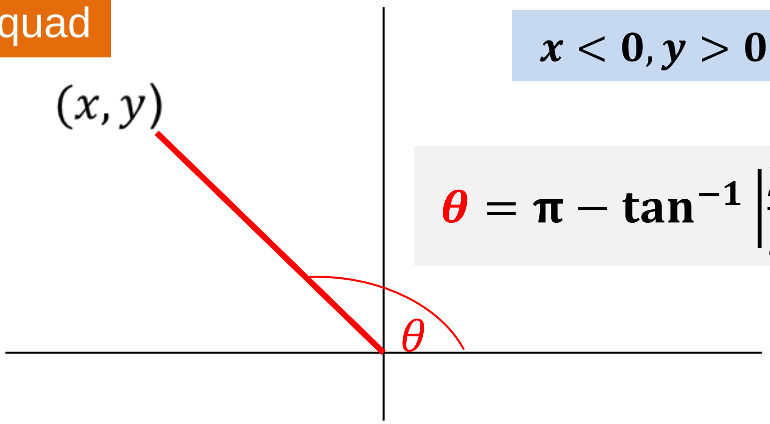
$$x, y > 0$$



$$\theta = \tan^{-1}\frac{y}{x}$$

2nd quad

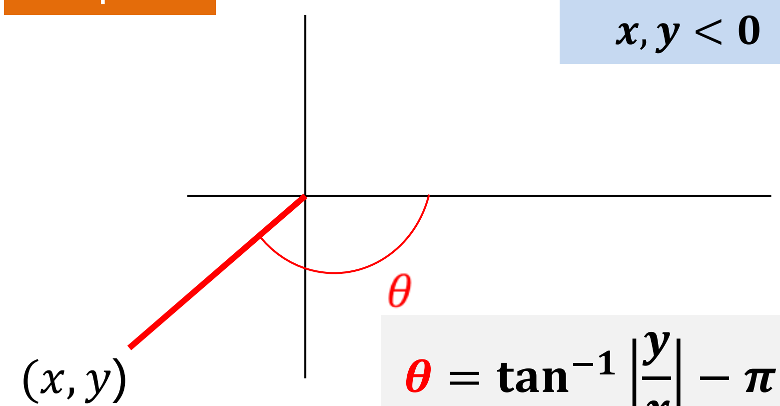
$$x < 0, y > 0$$



$$\theta = \pi - \tan^{-1}\left|\frac{y}{x}\right|$$

3rd quad

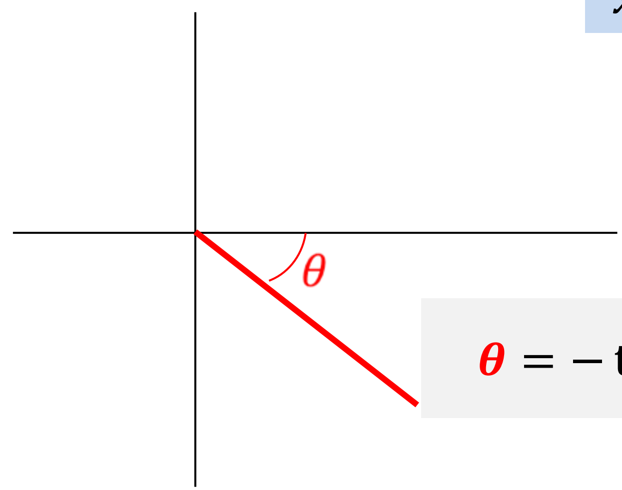
$$x, y < 0$$



$$\theta = \tan^{-1}\left|\frac{y}{x}\right| - \pi$$

4th quad

$$x > 0, y < 0$$



$$\theta = -\tan^{-1}\left|\frac{y}{x}\right|$$

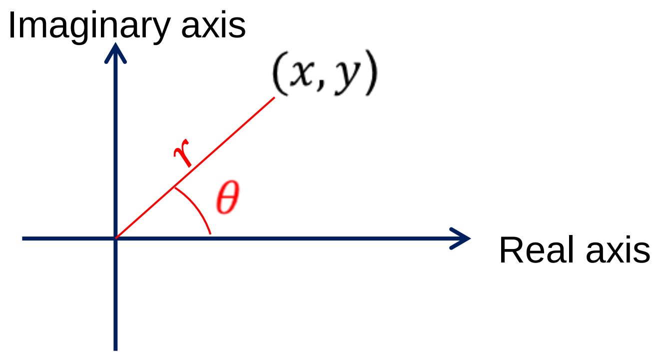
Complex Numbers: Properties of Amplitude

- $\text{amp}(\bar{z}) = -\text{amp}(z)$
- $\text{amp}(z_1 z_2) = \text{amp}(z_1) + \text{amp}(z_2) + 2\pi K \qquad \boldsymbol{K \in \{1, 0, -1\}}$
- $\text{amp}\left(\frac{z_1}{z_2}\right) = \text{amp}(z_1) - \text{amp}(z_2) + 2\pi K \qquad \boldsymbol{K \in \{1, 0, -1\}}$
- $\text{amp}(z^n) = n.\text{amp}(z) + 2\pi K \qquad \boldsymbol{K \in \{1, 0, -1\}}$
- $\text{amp}(z) = 0 \Rightarrow z \; is \; purely \; real \; \text{Re}(z) > 0$
- $\text{amp}(z) = \pi \Rightarrow z \; is \; purely \; real \; \text{Re}(z) < 0$
- $\text{amp}(z) = \frac{\pi}{2} \Rightarrow z \; is \; purely \; imaginary \; and \; img(z) > 0$
- $\text{amp}(z) = -\frac{\pi}{2} \Rightarrow z \; is \; purely \; imaginary \; and \; img(z) < 0$

Complex Numbers: Trigonometric Form or Polar Form

Every complex number $z = x + iy$ can be written in (r, θ) form i.e. polar form

$$z = r(\cos\theta + i\sin\theta)\ ;\ r = |z|\ ,\ \theta = amp(z)$$



Note : $\cos\theta + i\sin\theta$ is also denoted by $Cis(\theta)$

Example : $Cis\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

Complex Numbers: Properties of Polar Form

$$|\cos\theta + i\sin\theta| = 1 \qquad \text{if } \theta \in R$$

$$\overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$

$$Cis(\alpha)Cis(\beta) = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$= (\cos(\alpha+\beta) + i\sin(\alpha+\beta))$$

$$= Cis(\alpha+\beta)$$

$$\frac{Cis(\alpha)Cis(\beta)}{Cis(\gamma)Cis(\delta)} = Cis(\alpha+\beta-\gamma-\delta)$$

$$(Cis(\alpha))^n = Cis(n\alpha)$$

Complex Numbers: Exponential Form

$$z = \boldsymbol{re^{i\theta}}$$

Where $\mathrm{r} = |z|$ and θ is amplitude of z

$$\boldsymbol{e^{i\theta}} = \mathbf{1} + \frac{\boldsymbol{i\theta}}{1} + \frac{(\boldsymbol{i\theta})^2}{\mathbf{2}!} + \frac{(\boldsymbol{i\theta})^3}{3!} + \frac{(\boldsymbol{i\theta})^4}{4!} \ldots\ldots\ldots\ldots\ldots \infty$$

$$\boldsymbol{e^{i\theta}} = (\mathbf{1} - \frac{(\boldsymbol{\theta})^2}{\mathbf{2}!} + \frac{(\boldsymbol{\theta})^4}{4!} \ldots\ldots \infty) + \boldsymbol{i}(\boldsymbol{\theta} - \frac{(\boldsymbol{\theta})^3}{\mathbf{3}!} + \frac{(\boldsymbol{\theta})^5}{\mathbf{5}!} \ldots\ldots \infty)$$

$$\boldsymbol{e^{i\theta}} = \cos\theta + i\sin\theta$$

Complex Numbers: De Moivre's Theorem

$if\ \ n \in I \quad (\cos(\theta) + i\sin(\theta))^n = \cos(n\,\theta) + i\sin(n\theta)$

$if\ \ n\ is\ of\ type\ \frac{p}{q}, \mathrm{p}, \mathrm{q} \in I\ and\ \ \mathrm{q} \neq 0$ then

Complex Numbers: Cube Root of Unity

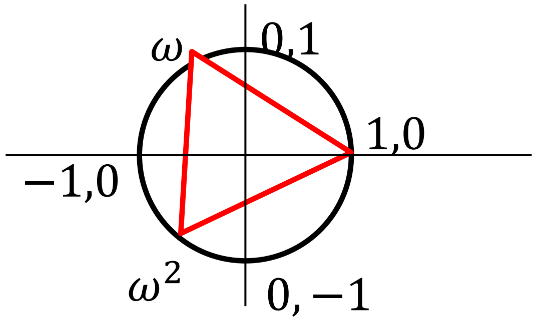
$$x = (1)^{\frac{1}{3}}$$

$$= x^3 - 1^3 = 0$$

$$= (x-1)(x^2+x+1) = 0$$

$$= x = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$$

$x = 1$, $\boldsymbol{\omega}$, $\boldsymbol{\omega^2}$ are cube roots of unity



NOTE: cube root of unity lies on vertices of an equilateral triangle inscribed in a unit circle having Centre at origin.

Complex Numbers: Properties of Cube Root of Unity

- $\omega^{3n} = 1, \quad \omega^{3n+1} = \omega, \quad \omega^{3n+2} = \omega^2$
- $1 + \omega + \omega^2 = 0$
- $\omega^{n+1} + \omega^{n+2} + \omega^{n+3} = 0 \;\; n \in I$
- $\frac{1}{\omega} = \omega^2, \;\; \frac{1}{\omega^2} = \omega$
- $\bar{\omega} = \omega^2$ and $\overline{\omega^2} = \omega$
- $\omega = e^{\frac{i2\pi}{3}} \quad \omega^2 = e^{-\frac{i2\pi}{3}}$

Complex Numbers: N^{th} Root of Unity

$$z^n = 1$$

$$z = (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{n}}$$

$$1 = \cos 2k\pi + i \sin 2k\pi \quad k \in I$$

$$z = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n} \qquad k = (0, 1, \ldots \ldots \ldots n-1)$$

$$z = 1, \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i \sin\frac{4\pi}{n} \ldots \ldots \cos\left(\frac{n-1}{n}\right)\pi + i \sin\left(\frac{n-1}{n}\right)\pi$$

$$z = 1, \alpha, \alpha^2, \alpha^3, \ldots \ldots \alpha^{n-1}$$

N^{th} root of unity are in G.P.

$$\alpha = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$$

Complex Numbers: Sum of n^{th} Root of Unity

- Sum of n^{th} root of unity is always $\boldsymbol{zero}$ i.e.

$$1+\alpha+\alpha^2+\alpha^3+\cdots..+\alpha^{n-1}=0 \because \frac{1-\alpha^n}{1-\alpha}=\frac{1-(\cos 2k\pi+i\sin 2k\pi)}{1-\alpha}=0$$

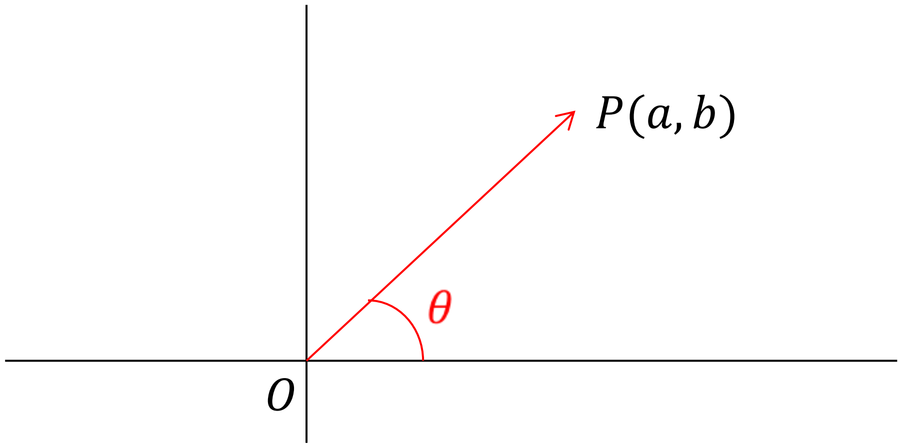
Where $\alpha=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$

All roots lie on circle centred at origin with radius $r^{\frac{1}{n}}$

All roots have same modulus and angle difference is $\frac{2\pi}{n}$

Complex Numbers: Vector Representation

$$z = a + ib$$



$$\overrightarrow{OP} = a\hat{\imath} + b\hat{\jmath}$$

$$\left|\overrightarrow{OP}\right| = \sqrt{a^2 + b^2}$$

Direction of **vector** $\overrightarrow{\boldsymbol{OP}} = \mathbf{arg}(\boldsymbol{z}) = \mathbf{tan}^{-1}\left(\frac{\boldsymbol{b}}{\boldsymbol{a}}\right)$

Complex Numbers: Rotation Theorem

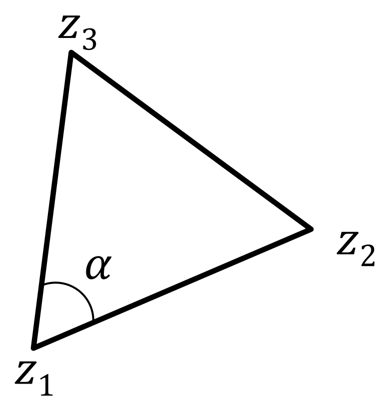
If a complex number z_2 is rotated by an α angle in **anti-clockwise** with respect to z_1 then relation between z_1, z_2, z_3 and α is given by ,

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}$$

Direction of $z_1 - z_2$ vector is towards z_1 from z_2

$$\frac{final\ vector}{initial\ vector} = \left|\frac{final\ vector}{initial\ vector}\right| e^{i\alpha}$$

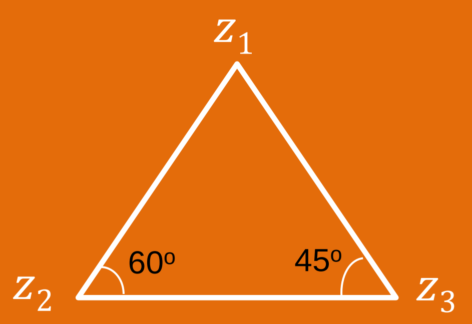
To find the angle of rotation choose a common point to rotate initial vector in direction of final vector



$$\frac{z_3 - z_2}{z_1 - z_2} = \frac{|z_3 - z_2|}{|z_1 - z_2|} e^{i\left(-\frac{\pi}{3}\right)}$$

$$\frac{z_3 - z_1}{z_3 - z_2} = \frac{|z_2 - z_1|}{|z_3 - z_2|} e^{i\left(-\frac{\pi}{4}\right)}$$

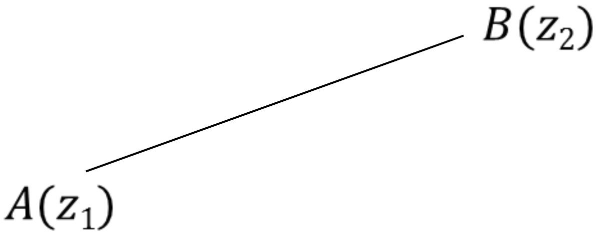
Negative angle because of clockwise rotation



Complex Numbers: Geometrical Results

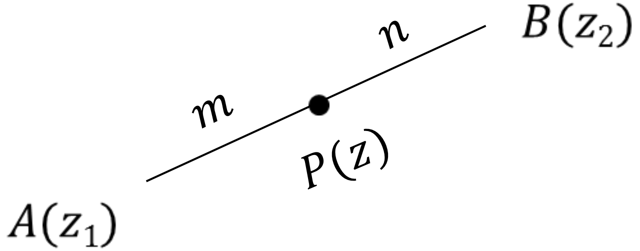
Distance formula

$$AB = |z_2 - z_1|$$



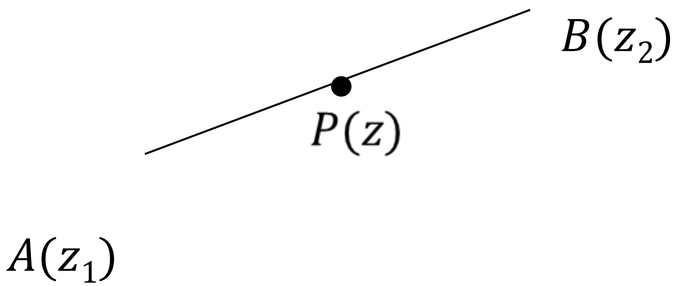
Section formula

$$z = \frac{mz_2 + nz_1}{m + n}$$



Mid-point formula

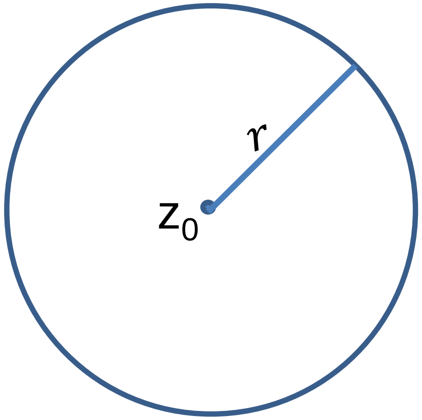
$$z = \frac{z_1 + z_2}{2}$$



Complex Number: Position of Point

Equation of circle with centre z_0 and radius r

$$|\boldsymbol{z} - \boldsymbol{z_0}| = \boldsymbol{r}$$



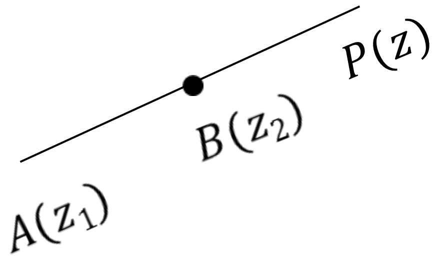
1. $|z_1 - z_0| < r \Rightarrow z_1$ lies inside the circle
2. $|z_1 - z_0| > r \Rightarrow z_1$ lies outside the circle
3. $|z_1 - z_0| = r \Rightarrow z_1$ lies on the circle

Points locations depends on distance from Centre

Complex Numbers: Straight Line in Different Form

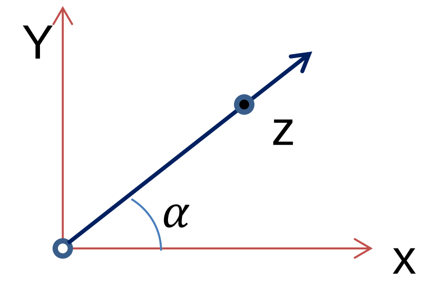
1. amp$\left(\frac{z-z_1}{z_2-z_1}\right) = 0$ since $\lambda \in R$

2. $\frac{z-z_1}{z_2-z_1} = \frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$

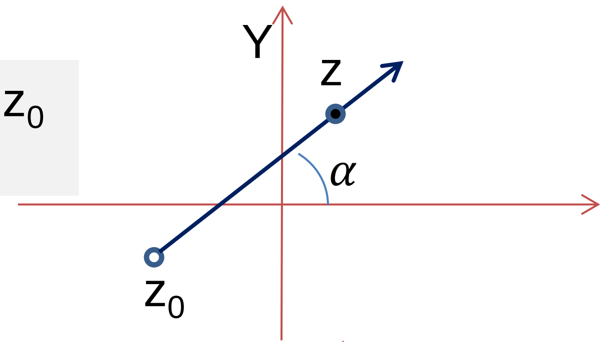


Complex Numbers: General Locus of z

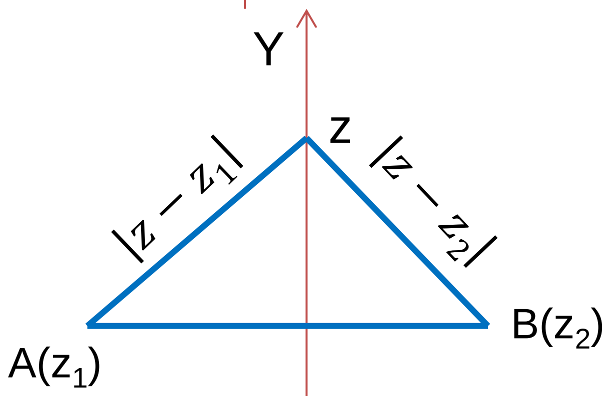
$\arg(z) = \alpha \rightarrow$ Represent a ray passing through origin (excluded origin) having slope α



$\arg(z - z_0) = \alpha \rightarrow$ represent a ray starting from z_0 excluded z_0 and slope α



$|z - z_1| = |z - z_2|$
$\rightarrow$ *perpendicular bisector of AB.* $A(z_1) \& B(z_2)$



Complex Numbers: General Locus of z

$$\frac{|z - z_1|}{|z - z_2|} = k \;\rightarrow k \in (0,1)u(1,\infty)\;\; represent\ circle$$

$$|\boldsymbol{z - z_1}| + |\boldsymbol{z - z_2}| = \boldsymbol{k}$$

$if\ k > |z_1 - z_2|$ represent ellipse with foci $at\ z_1\, and\ z_2$ & length of major axis is k

If $k\ = |z_1 - z_2|$represents line

If $k < |z_1 - z_2|$does not represents anything

Complex Numbers: General Locus

$$|z - z_1| - |z - z_2| = k$$

If $k > 0$ & $k < |z_1 - z_2|$, then it represents a hyperbola with foci z_1 & z_2

If $k = |z_1 - z_2|$, then it represents a straight line passing through z_1 & z_2 and excluding segment z_1, z_2