Differential Equations

By
Ankush Garg(B. Tech, IIT Jodhpur)

DIFFERENTIAL EQUATION :

An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **differential equation**.

degree = 1
order = 1 → $\boxed{\left(\frac{dy}{dx}\right)^1 = x + y}$

degree = 1 ← $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = x$

ORDER OF DIFFERENTIAL EQUATION :

The order of a differential equation is the order of the highest differential coefficient occurring in it.

DEGREE OF DIFFERENTIAL EQUATION :

(Power)

The exponent of the highest order differential coefficient, when the differential equation is expressed as a polynomial in all the differential coefficient.

Thus the differential equation :

$$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \ldots\ldots\ldots = 0$$ is of order m & degree p.

Note :

(i) The exponents of all the differential coefficient should be free from radicals and fraction.

(ii) The degree is always positive natural number.

(iii) The degree of differential equation may or may not exist.

Find the order and degree of the following differential equation :

(i) $\sqrt[3]{\frac{d^2y}{dx^2}} = \sqrt{\frac{dy}{dx} + 3}$ (ii) $\frac{d^2y}{dx^2} = \sin\left(\frac{dy}{dx}\right)$ (iii) $\frac{dy}{dx} = \sqrt{3x+5}$

$$\left(\frac{d^2y}{dx^2}\right)^{3} = \left(\frac{dy}{dx}+3\right)^2$$

order = 2,
Degree = 3

order = 2
Degree = D.N.E

order = 1
degree = 2

FORMATION OF A DIFFERENTIAL EQUATION :

If an equation in independent and dependent variables having some arbitrary constant is given then a differential equation is obtained as follows :

- Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it .
- Eliminate the arbitrary constants .

The eliminant is the required differential equation . Consider forming a differential equation for $y^2 = 4a(x+b)$ where a and b are arbitary constant .

$$y^2 = 4a(x+b)$$

$$2yy' = 4a$$

$$2yy'' + 2y'y' = 0$$

$$2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 0$$

$$y = mx$$

$$\boxed{\frac{dy}{dx} = m}$$

$$\boxed{y = \left(\frac{dy}{dx}\right)x}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$y = 2x$ fixed constant

$\boxed{y = mx}$ arbitrary (constant)

SOLUTION OF DIFFERENTIAL EQUATION :

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives, but satisfying the given differential equation (i.e., from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx} = e^x$ could be obtained by simply integrating both sides, i.e., $y = e^x + c$

$$\frac{dy}{dx} = e^x$$

$$\int dy = \int e^x dx$$

$$\boxed{y = e^x + c}$$

solution

$$\boxed{\frac{dy}{dx} = \frac{y}{x}}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + \ln c$$

$$\ln y = \ln cx$$

$$\boxed{y = mx}$$

$$\left(\frac{dy}{dx}\right)^1$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \times$$

ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS :

(a) Separation of Variables :

Some differential equations can be solved by the method of separation of variables (or "variable separable"). This method is only possible, if we can express the differential equation in the form

$$A(x)dx + B(y)\,dy = 0$$

where A(x) is a function of 'x' only and B(y) is a function of 'y' only.

A general solution of this is given by,

$$\int A(x)\,dx + \int B(y)dy = c$$

where 'c' is the arbitrary constant.

$$\int A(x)\,dx = \int B(y)\,dy$$

Solve the differential equation $xy\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}(1 + x + x^2)$.

$$\int \frac{y\,dy}{1+y^2} = \int \frac{dx}{x(1+x^2)}(1+x+x^2)$$

$$= \int \frac{[(1+x^2)+x]\,dx}{(1+x^2)\,x}$$

$$\int \frac{y\,dy}{1+y^2} = \int \frac{dx}{x} + \int \frac{dx}{1+x^2}$$

Differential Equations Reducible to the Separable Variable Type :

Sometimes differential equation of the first order cannot be solved directly by variable separation. By some substitution we can reduce it to a differential equation with separable variables. A differential equation of the form $\frac{dy}{dx} = f(ax+by+c)$ is solved by writing $ax + by + c = t$.

Solve $(x-y)^2 \frac{dy}{dx} = 1$.

$$x - y = t$$

$$1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{dt}{dx}$$

$$t^2 \cdot \left(1 - \frac{dt}{dx}\right) = 1$$

$$1 - \frac{dt}{dx} = \frac{1}{t^2}$$

$$-\frac{dt}{dx} = \frac{1}{t^2} - 1$$

$$-\frac{dt}{dx} = \frac{1-t^2}{t^2}$$

$$\int \frac{-t^2}{1-t^2}\, dt = \int dx$$

② Homogeneous Equation

When $\frac{dy}{dx}$ is equal to a fraction whose numerator and the denominator both are homogeneous functions of x and y of the same degree then the differential equation is said to be homogeneous equation.

i.e. when $\frac{dy}{dx} = f(x, y)$, where f(km, ky) = f(x, y) then this differential equation is said to be homogeneous differential equation .

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

Solve $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$.

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{y}\right) + \frac{1}{2}\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{x^2 + y^2 + 2xy}{x^2}$$

$$\boxed{\frac{dy}{dx} = f(y/x)}$$

$$\frac{dy}{dx} = 1 + (y/x)^2 + 2(y/x)$$

$$\frac{y}{x} = t \Rightarrow y = tx$$

① $\boxed{y = tx}$

$$\frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$t + x\frac{dt}{dx} = \frac{1}{2}\left[\frac{1}{t} + t\right]$$

$$t + x\frac{dt}{dx} = \frac{t^2+1}{2t}$$

$$t + x\frac{dt}{dx} = \frac{t^2+1}{2t}$$

$$x\frac{dt}{dt} = \frac{t^2+1}{2t} - t$$

$$x\frac{dt}{dt} = \frac{t^2+1-2t^2}{2t}$$

$$\boxed{x\frac{dt}{dx} = \frac{1-t^2}{2t}}$$

$$\int \frac{2t\,dt}{1-t^2} = \int \frac{dx}{x}$$

Equations reducible to homgeneous form

The equation of the form $\frac{dy}{dx} = \frac{(a_1x + b_1y + c_1)}{(a_2x + b_2y + c_2)}$ where $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

can be reduced to homgeneous form by changing the variable x, y to u,v as

$x = u + h$, $y = v + k$

The equation becomes, $\frac{dv}{du} = \frac{(a_1u + b_1v) + (a_1h + b_1k + c_1)}{(a_2u + b_2v) + (a_2h + b_2k + c_2)}$

$$\frac{dv}{du} = \frac{a_1 + (b_1)(v/u)}{a_2 + b_2(v/u)}$$

$a_1h + b_1k + c_1 = 0$

$a_2h + b_2k + c_2 = 0$

$h, k = ?$

$$\boxed{\frac{v}{u} = t}$$

60% — 70%

3 LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER :

The most general form of a linear differential equations of first order is $\boxed{\frac{dy}{dx} + Py = Q}$, where P & Q are functions of x.

To solve such an equation multiply both sides by $e^{\int P dx}$.

$$\boxed{\frac{dy}{dx} + P(x)\,y = g(x)}$$

① $I.F = e^{\int P(x)\,dx}$

Sol^n ② $\boxed{y \times I.F = \int g(x) \times I.F}$

9) Solve $\frac{dy}{dx} + by = e^{nx}$.

① $I.F = e^{\int b\,dx}$

$= e^{bx}$

② $y \times e^{bx} = \int e^{nx} \times e^{bx}\, dx$

$y e^{bx} = \int e^{(n+b)x}\, dx$

$x=0 \quad y=1$

$(0, 1)$

$$\boxed{y \cdot e^{bx} = \frac{e^{(n+b)x}}{(n+b)} + C}$$

$\Rightarrow 1 = \frac{1}{(n+b)} + C$

$\boxed{1 - \frac{1}{(n+b)} + C}$

SAFALTA.COM

9) $$\frac{dy}{dx}+\frac{y}{(1+x^2)}=\frac{e^{\tan^{-1}x}}{1+x^2}$$

$$\frac{dy}{dx}+\left(\frac{-1}{1+x^2}\right)y=\left(\frac{e^{\tan^{-1}x}}{1+x^2}\right)$$

$$I.F=e^{\int\frac{-1}{1+x^2}dx}$$

$$I.F=e^{\tan^{-1}x}$$

$$y\times e^{\tan^{-1}x}=\int\left(\frac{e^{\tan^{-1}x}}{1+x^2}\right)\times\left(e^{\tan^{-1}x}\right)dx$$

$$\boxed{y\times e^{\tan^{-1}x}=\int\frac{e^{2\tan^{-1}x}}{(1+x^2)}dx}$$

4

Bernoulli's Equation:

(Eqⁿ Reducible to Linear)

$$\frac{dy}{dx} + Py = Qy^n \qquad (1)$$

where P and Q are function of x alone or constants. Divide each term of (1) by y^n . We get

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q \qquad (2)$$

Let $\frac{1}{y^{n-1}} = v$, so that $\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx}$

Substituting in (2) , we get

$$\boxed{\frac{dv}{dx} + (1-n)vP = Q(1-n)} \qquad (3)$$

(3) is a linear differential equation

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P}{y^{n-1}} = Q$$

$$\frac{1}{y^{n-1}} = v$$

$$\frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dv}{dx}$$

$=$

Solve the differential equation $x\dfrac{dy}{dx}+y=x^3y^6$.

$$\frac{dy}{dx}+\frac{y}{x}=x^2y^6$$

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{x}\frac{1}{y^5}=x^2$$

$$\frac{1}{y^5}=t$$

$$\frac{-5}{y^6}\frac{dy}{dx}=\frac{dt}{dx}$$

$$-\frac{1}{5}\frac{dt}{dx}+\frac{1}{x}t=x^2$$

$$\frac{dt}{dx}\left(-\frac{5}{x}\right)t=-5x^2$$

$$I.F=e^{\int -5/x\,dx}$$

$$t\times I.F=\int -5x^2\times I.F\,dx$$

Topic

ORTHOGONAL TRAJECTORY

Any curve, which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family. For example, each straight line passing through the origin, i.e. $y = kx$ is an orthogonal trajectory of the family of the circles $x^2 + y^2 = a^2$

Procedure for finding the orthogonal trajectory

(i) Let $f(x, y, c) = 0$ be the equation of the given family of curves, where c is an arbitrary parameter.

(ii) Differentiate $f = 0$, w.r.t. x and eliminate c i.e. form a differential equation.

(iii) Substitute $\boxed{-\frac{dx}{dy} \text{ for } \frac{dy}{dx}}$ in the above differential equation. This will give the differential equation of the orthogonal trajectory.

Ex→ $(x^2+y^2-a^2=0)$

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\left(\frac{dy}{dx}\right) = 0$$

$$x + y\left(-\frac{dx}{dy}\right) = 0$$

$$x = y\frac{dx}{dy}$$

$$\int\frac{dy}{y} = \int\frac{dx}{x} \Rightarrow \ln y = \ln x + \ln c$$

$$\ln y = \ln(cx)$$

$$\boxed{y = cx}$$

$\ln x$

e^x

SAFALTA.COM

9) Find the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 - y = c$.

① $x^2+2y^2-y-c=0$

② $2x+4y\frac{dy}{dx}-\frac{dy}{dx}=0$

③ $2x+4y\left(-\frac{dx}{dy}\right)-\left(-\frac{dx}{dy}\right)=0$

$2x+4y\left(-\frac{dx}{dy}\right)+\frac{dx}{dy}=0$

$2x\frac{dy}{dx}+(1-4y)=0$

$2x\frac{dy}{dx}=(4y-1)$

$\Rightarrow \int\frac{dy}{4y-1}=\int\frac{dx}{2x}$

Geometrical applications :

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at point P is $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$

(i) The equation of the tangent at P is $\boxed{y - y_1 = \frac{dy}{dx}(x - x_1)}$

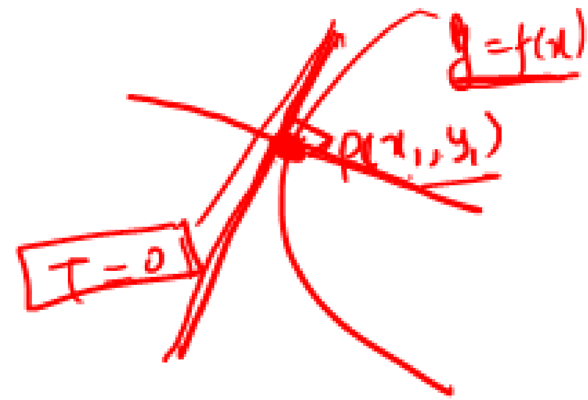
y=0

~~x-intercept~~ of the tangent $= x_1 - y_1\left(\frac{dx}{dy}\right)$

x=0

~~y-intercept~~ of the tangent $= y_1 - x_1\left(\frac{dy}{dx}\right)$

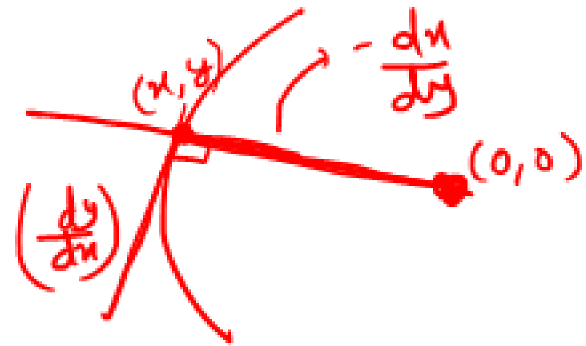
(ii) The equation of normal at P is $y - y_1 = -\frac{1}{(dy/dx)}(x - x_1)$

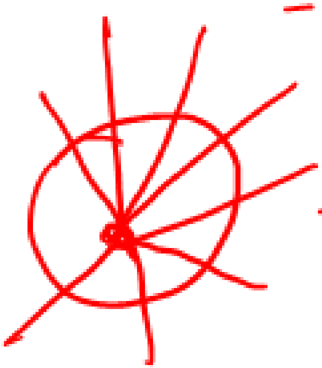


$m \longrightarrow -\frac{1}{m}$

Q. Find the equation of the curve for which the normal at any point (x,y) passes through the origin.

$$\boxed{\underset{0}{Y} - y = -\frac{dx}{dy}(\underset{0}{X} - x)}$$





$$-y = -\frac{dx}{dy}(-x)$$

$$-y = \frac{x\,dx}{dy}$$

$$-\int y\,dy = \int x\,dx$$

$$-\frac{y^2}{2} = \frac{x^2}{2} + C \Rightarrow \boxed{\frac{x^2}{2} + \frac{y^2}{2} + c = 0}$$

$$\boxed{x^2 + y^2 = a^2}$$

Exact Differential Equation

For this the following results must be memorized.

(i) $d(x + y) = dx + dy$

(ii) $d(xy) = y\,dx + x\,dy$

(iii) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

(iv) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

(v) $d(\log xy) = \frac{y\,dx + x\,dy}{xy}$

(vi) $d\left(\log \frac{y}{x}\right) = \frac{(x\,dy - y\,dx)}{xy}$

(vii) $d\left(\frac{1}{2}\log \frac{x+y}{x-y}\right) = \frac{x\,dy - y\,dx}{x^2 - y^2}$

(viii) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2 + y^2}$

(ix) $\frac{d\left[f(x,y)\right]^{1-n}}{1-n} = \frac{f'(x,y)}{(f(x,y))^n}$

(x) $d\left(\sqrt{x^2 + y^2}\right) = \frac{x\,dx + y\,dx}{\sqrt{x^2 + y^2}}$