Circles

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Concepts

EQUATION OF A CIRCLE IN VARIOUS FORM:

$(x-h)^2+(y-k)^2=r^2$

(a) The circle with centre (h, k) & radius 'r' has the equation;
$(x-h)^2 + (y-k)^2 = r^2$.

(b) The general equation of a circle is $\boxed{x^2+y^2+2gx+2fy+c=0}$ with centre as:
$(-g, -f)$ & radius $= \sqrt{g^2+f^2-c}$.

$\rightarrow c(-g,-f)$ $\quad r=\sqrt{g^2+f^2-c}$

Remember that every second degree equation in x & y in which coefficient of x^2 = coefficient of y^2 & there is no xy term always represents a circle.

If $g^2+f^2-c>0 \Rightarrow$ real circle.
$g^2+f^2-c=0 \Rightarrow$ point circle.
$g^2+f^2-c<0 \Rightarrow$ imaginary circle.

$ax^2+2hxy+by^2+2gx+2fy+c=0$

$\boxed{a=b \quad h=0}$ For circle.

Note that the general equation of a circle contains three arbitrary constants, g, f & c which corresponds to the fact that a unique circle passes through three non collinear points.

(c) The equation of circle with (x_1, y_1) & (x_2, y_2) as its diameter is :
$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0$.

Note that this will be the circle of least radius passing through (x_1, y_1) & (x_2, y_2).

Problems

If $y = 2x + m$ is a diameter to the circle $x^2 + y^2 + 3x + 4y - 1 = 0$, then find m

$C(-g, -f)$

$C\left(-\frac{3}{2}, -2\right)$

$-2 = 2\left(-\frac{3}{2}\right) + m$

$-2 = -3 + m$

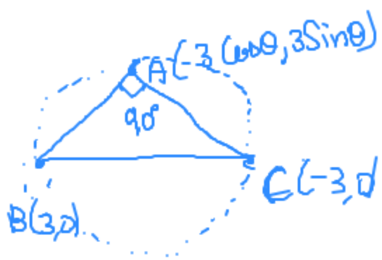
$\boxed{m = 1}$

Problems

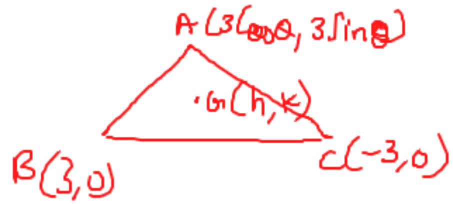
$x^2+y^2=a^2$ $(a\cos\theta, a\sin\theta)$

B and C are fixed points having co–ordinates (3, 0) and (– 3, 0) respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation :

(A) $x^2 + y^2 = 1$ (B) $x^2 + y^2 = 2$ (C) $9(x^2 + y^2) = 1$ (D) $9(x^2 + y^2) = 4$



A is moving such that $\angle BAC = 90^\circ$



eqn of circle

$(x-3)(x+3)+(y-0)(y-0)=0$

$x^2-9+y^2=0$

$\boxed{x^2+y^2=9}$ → A lies on this $(3\cos\theta, 3\sin\theta)$

$h = \frac{3\cos\theta+3-3}{3} = \cos\theta$

$k = \frac{3\sin\theta+0+0}{3} = \sin\theta$

$\sin^2\theta+\cos^2\theta=1 \Rightarrow \boxed{h^2+k^2=1}$

$x^2+y^2=1$

INTERCEPTS MADE BY A CIRCLE ON THE AXES :

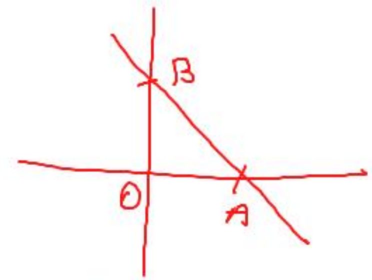
The intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ on the co-ordinate axes are $\boxed{2\sqrt{g^2 - c}}$ & $\boxed{2\sqrt{f^2 - c}}$ respectively.

NOTE :

If $g^2 - c > 0$ $\Rightarrow$ circle cuts the x axis at two distinct points.

If $g^2 = c$ $\Rightarrow$ circle touches the x-axis.

If $g^2 < c$ $\Rightarrow$ circle lies completely above or below the x-axis.



x-intercept = OA

y- '' = OB

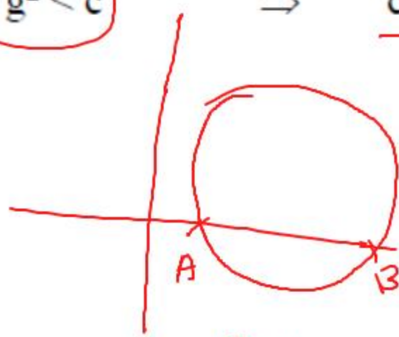
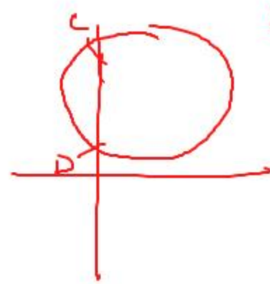
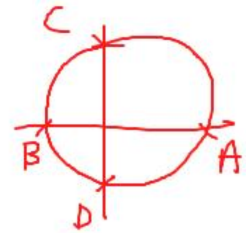


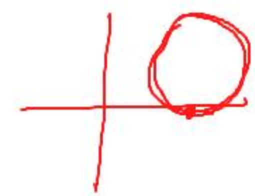
Fig 1

AB = x-intercept

CD = y-intercept.







Problems

The equation of a circle whose centre is (3, –1) and which cuts off a chord of length 6 on the line $2x - 5y + 18 = 0$

(A) $(x-3)^2 + (y+1)^2 = 38$

(B) $(x+3)^2 + (y-1)^2 = 38$

(C) $(x-3)^2 + (y+1)^2 = \sqrt{38}$

(D) none of these

$p = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}} \right|$

Centre (3, –1)

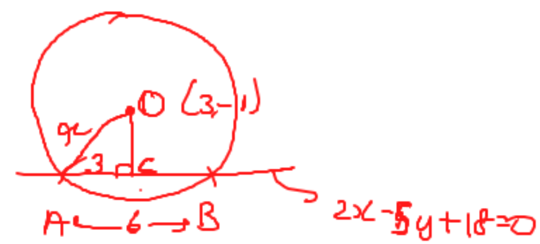
Radius $\sqrt{38}$

$(x-3)^2 + (y+1)^2 = 38.$

AC = 3

OA = r (To find)

OC = length of ⊥ from O on given line



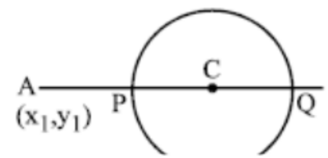
$\left| \frac{6+5+18}{\sqrt{29}} \right| = \frac{29}{\sqrt{29}} = \sqrt{29} = OC$

$r^2 = OC^2 + AC^2 \Rightarrow r^2 = 29 + 9 \Rightarrow r = \sqrt{38}$

POSITION OF A POINT w.r.t. A CIRCLE :

The point (x_1, y_1) is inside, on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \Leftrightarrow 0$.

Note : The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC – r respectively.



eg) $x^2+y^2-3x+2y+6=0$ $(3,2)$

S: $ax^2+2hxy+by^2+2gx+2fy+c$

If I substitute $P(x_1,y_1)$

S_1: $ax_1^2+2hx_1y_1+by_1^2+2gx_1+2fy_1+c$

T: $axx_1+h(x_1y+xy_1)+byy_1+g(x+x_1)+f(y+y_1)+c$

S: $x^2+y^2-3x+2y+6$

S_1: $9+4-9+4+6=14$

Replace.
$x^2 \rightarrow xx_1$
$y^2 \rightarrow yy_1$
$2x \rightarrow x+x_1$
$2y \rightarrow y+y_1$
$2xy \rightarrow x_1y+xy_1$

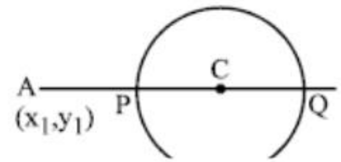
Rules for writing T at x_1, y_1

POSITION OF A POINT w.r.t. A CIRCLE :

The point (x_1 , y_1) is inside, on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \Leftrightarrow 0$.

$< = > 0$

Note : The greatest & the least distance of a point A from a circle with centre C & radius r is AC + r & AC – r respectively.



3 possibilities

→ Point inside the circle: $x_1^2+y_1^2+2gx_1+2fy_1+c < 0$

→ Point outside the circle. $x_1^2+y_1^2+2gx_1+2fy_1+c > 0$

on the circle $x_1^2+y_1^2+2gx_1+2fy_1+c = 0$

$\cdot A(x_1,y_1)$

$x^2+y^2+2gx+2fy+c=0$

inside:	$S_1 < 0$
outside:	$S_1 > 0$
on	$S_1 = 0$

Eg) $x^2+y^2-2x-4y+1=0$

$P(3,0)$

$S: x^2+y^2-2x-4y+1$

$S_1: 9-6+1 = 4 > 0$

outside the circle.

Problems

If P(2, 8) is an interior point of a circle $x^2 + y^2 - 2x + 4y - p = 0$ which neither touches nor intersects the axes, then set for p is -

(A) $p < -1$ (B) $p < -4$ (C) $p > 96$ (D) ϕ ✓

$S: x^2+y^2-2x+4y-p$

$g=-1,\ f=2,\ c=-p$

① $g^2<c$

$1<-p \Rightarrow -p>1 \Rightarrow \boxed{p<-1}$ — ①

② $f^2<c$

$4<-p \Rightarrow -p>4 \Rightarrow \boxed{p<-4}$ — ②

③ $S_1<0$

$S_1 = 4+64-4+32-p = 96-p$

$96-p<0 \Rightarrow \boxed{p>96}$ — ③

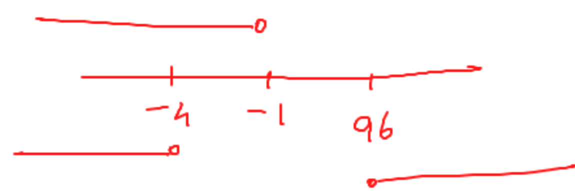
neither touches nor intersects x-axis

" " " " y-axis

P(2,8) is an interior point

$2\sqrt{g^2-c} \Rightarrow \boxed{\begin{array}{l} g^2-c<0 \\ f^2-c<0 \\ S_1<0 \end{array}}$

① ∩ ② ∩ ③

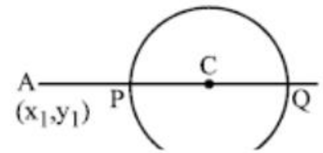


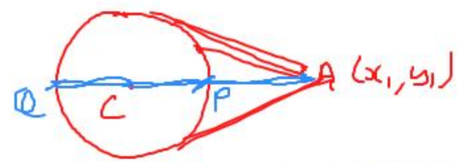
No common values.

POSITION OF A POINT w.r.t. A CIRCLE :

The point (x_1, y_1) is inside, on or outside the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. according as $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \Leftrightarrow 0$.

Note : The greatest & the least distance of a point A from a circle with centre C & radius r is $AC + r$ & $AC - r$ respectively.





Min distance of $A(x_1, y_1)$ from circle $= AP$

Max distance $= AQ$

$AP = AC - r$

$AQ = AC + r.$

LINE & A CIRCLE :

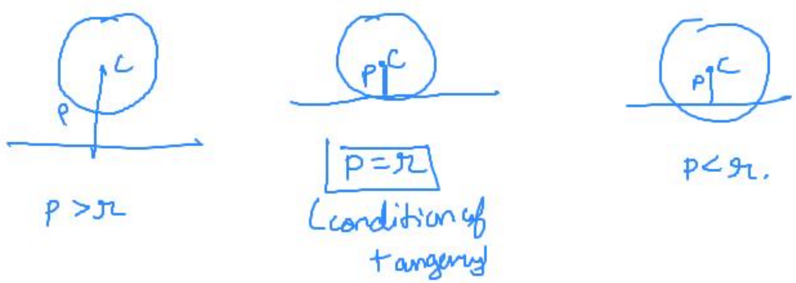
Let $L = 0$ be a line & $S = 0$ be a circle. If r is the radius of the circle & p is the length of the perpendicular from the centre on the line, then :

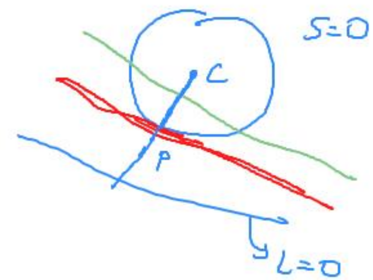
(i) $p > r \Leftrightarrow$ the line does not meet the circle i. e. passes out side the circle.

(ii) $p = r \Leftrightarrow$ the line touches the circle.

(iii) $p < r \Leftrightarrow$ the line is a secant of the circle.

(iv) $p = 0 \Rightarrow$ the line is a diameter of the circle.





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TANGENT & NORMAL :

(a) The equation of the tangent to the circle $x^2+y^2=a^2$ at its point (x_1, y_1) is, $x x_1 + y y_1 = a^2$. Hence equation of a tangent at $(a\cos\alpha, a\sin\alpha)$ is ; $x\cos\alpha + y\sin\alpha = a$. The point of intersection of the tangents at the points $P(\alpha)$ and $Q(\beta)$ is

$$\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{a\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

Point Form

Case-1

$x^2+y^2=a^2$ $\quad P(x_1, y_1)$

Eqn of Tangent is $T=0$

$\boxed{xx_1+yy_1=a^2}$

(b) The equation of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ at its point (x_1, y_1) is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

$T=0 \Rightarrow xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

(c) $y=mx+c$ is always a tangent to the circle $x^2+y^2=a^2$ if $c^2=a^2(1+m^2)$ and the point of contact is $\left(-\frac{a^2m}{c}, \frac{a^2}{c}\right)$.

$x^2+y^2=a^2$

$y=mx+c \Rightarrow c^2=a^2(1+m^2) \Rightarrow c=\pm a\sqrt{1+m^2}$

$\boxed{\text{Eqn of tangent is } \quad y=mx\pm a\sqrt{1+m^2}}$ Slope Form

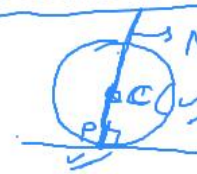
(d) If a line is normal/orthogonal to a circle then it must pass through the centre of the circle. Using this fact normal to the circle $x^2+y^2+2gx+2fy+c=0$ at (x_1, y_1) is

$$y-y_1=\frac{y_1+f}{x_1+g}(x-x_1).$$

$P(x_1, y_1)$ $\quad C(-g,-f)$

Two point form

Normal at P



Problems

If the straight line $ax + by = 2$; $a, b \neq 0$ touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are respectively

(A) 1, –1 (B) 1, 2 (C) $-\frac{4}{3}$, 1 (D) 2, 1

$x^2+y^2-2x=3$

$r = \sqrt{1+0+3}$
$= 2$

(1,0)

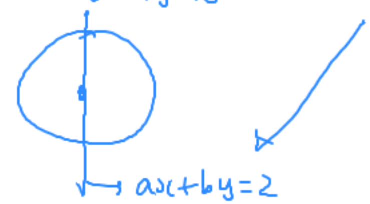
2

$ax+by=2$

Line

$ax+by=2$

$x^2+y^2-4y=6$ $C(0,2)$



$ax+by=2$

condition of tangency $p=r$

$p = \perp = \left|\frac{a+0-2}{\sqrt{a^2+b^2}}\right| = 2$

$|a-2| = 2\sqrt{a^2+1}$

$a^2+\cancel{4}-4a = 4a^2+\cancel{4} \Rightarrow 3a^2+4a=0$

$a(3a+4)=0 \Rightarrow a=0, -\frac{4}{3}$

$a(0)+b(2)=2$

$2b=2$

$b=1$

LENGTH OF A TANGENT AND POWER OF A POINT :

The length of a tangent from an external point (x_1, y_1) to the circle

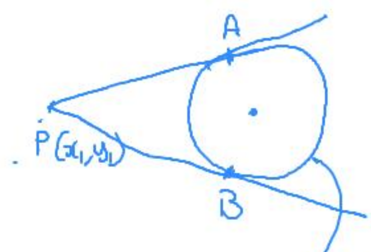
$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $L = \sqrt{x_1^{\,2} + y_1^{\,2} + 2gx_1 + 2f_1y + c} = \sqrt{S_1}$.

Square of length of the tangent from the point P is also called **THE POWER OF POINT** w.r.t. a circle. Power of a point remains constant w.r.t. a circle.

Note that : power of a point P is positive, negative or zero according as the point 'P' is outside, inside or on the circle respectively.

Length of tangent from (x_1, y_1) is $\sqrt{S_1}$

Power of a point $= (\sqrt{S_1})^2 = S_1$



Length of tangent = PA = PB

Formula = $\sqrt{S_1}$

$S: x^2 + y^2 + 2gx + 2fy + c = 0$

Problems

If L_1 and L_2 are the length of the tangent from (0, 5) to the circles $x^2 + y^2 + 2x - 4 = 0$ and $x^2 + y^2 - y + 1 = 0$ then

(A) $L_1 = 2L_2$ (B) $L_2 = 2L_1$ (C) $L_1 = L_2$ (D) $L_1^2 = L_2$

$L_1 = \sqrt{S_1} = \sqrt{0+25+0-4} = \sqrt{21}$

$L_2 = \sqrt{S_2} = \sqrt{0+25-5+1} = \sqrt{21}$

$L_1 = L_2$.

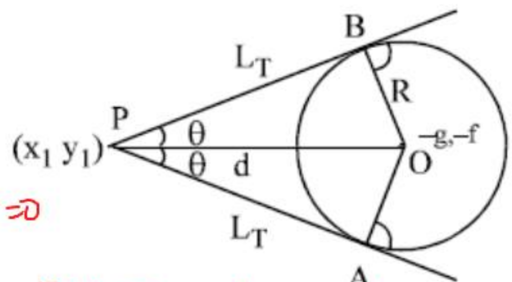
CHORD OF CONTACT :

If two tangents PT_1 & PT_2 are drawn from the point $P(x_1, y_1)$ to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, then the equation of the chord of contact T_1T_2 is :
$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$

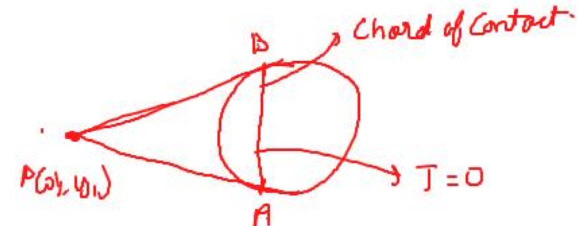
Eqn of C.O.C

$\boxed{T=0}$

$xx_1 + yy_1 + g(x+x_1) + f(y+y_1) + c = 0$



PA & PB are the tangents



PA = PB = Length of Tangent.

Chord of contact exists only if the point 'P' is not inside.

The joint equation of a pair of tangents drawn from the point $A(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is : $\boxed{SS_1 = T^2.}$ **

Where $S \equiv x^2 + y^2 + 2gx + 2fy + c$; $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

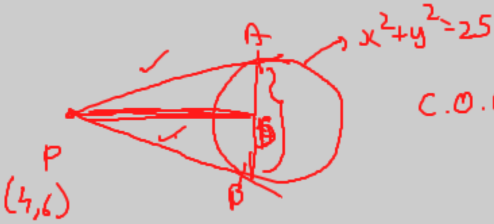
$T \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c.$

$$\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

Problems

(i) Find the equation of the chord of contact of the point (1, 2) with respect to the circle $x^2 + y^2 + 2x + 3y + 1 = 0$

(ii) Tangents are drawn from the point P(4, 6) to the circle $x^2 + y^2 = 25$. Find the area of the triangle formed by them and their chord of contact.



C.O.C = T=0

$4x+6y=25$

$$PD = \left|\frac{4\times4+6\times6-25}{\sqrt{4^2+6^2}}\right| = \frac{27}{\sqrt{52}} = \frac{27}{2\sqrt{13}}$$

Area of Δ PAB = 2× Area of ΔPAD

length of tangent = $\sqrt{S_1}$

$\sqrt{4^2+6^2-25} = \sqrt{27}$

$PA^2 = PD^2 + AD^2$

$27 = \frac{27^2}{52} + AD^2$

$AD^2 = 27 - \frac{27^2}{52} = \frac{27\times25}{52}$

$AD = \sqrt{\frac{27\times25}{52}}$

$$\text{Area}_{PAD} = \frac{1}{2}\times PD\times AD = \boxed{\frac{1}{\cancel{2}}\;\frac{27}{\sqrt{52}}\times\frac{\sqrt{27\times25}}{\sqrt{52}}\times 2}$$

×2

Ans

Problems

If $x = 3$ is the chord of contact of the circle $x^2 + y^2 = 81$, then the equation of the corresponding pair of tangents, is

(A) $x^2 - 8y^2 + 54x + 729 = 0$

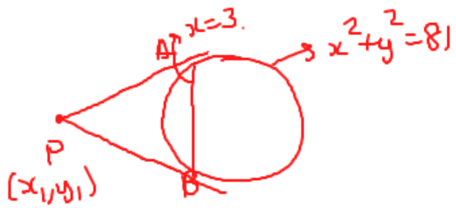
(B) $x^2 - 8y^2 - 54x + 729 = 0$

(C) $x^2 - 8y^2 - 54x - 729 = 0$

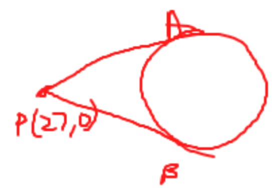
(D) $x^2 - 8y^2 = 729$

$T \Rightarrow 27x = 81$

$x = 3$



$p(27,0)$



PA & PB

$SS_1 = T^2$

$(x^2+y^2-81)(27^2-81) = (x-3)^2$

$(x^2+y^2-81)(648) = x^2 \cancel{+y} - 6x + 9$

$648x^2+648y^2-81\times 648 = x^2-6x+9.$

$647x^2+648y^2+6x - 81\times 648 - 9 = 0$

C.O.C $T=0$

$xx_1 + yy_1 = 81$ — 1

same lines

$x = 3$ — 2

$27x = 81$

$x_1 = 27,\ y_1 = 0$

EQUATION OF THE CHORD WITH A GIVEN MIDDLE POINT :

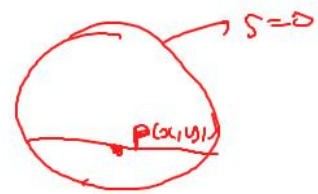
The equation of the chord of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ in terms of its mid point $M(x_1, y_1)$ is $\boxed{y - y_1 = -\frac{x_1 + g}{y_1 + f}(x - x_1)}$. This on simplication can be put in the form

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

which is designated by $\boxed{T = S_1}$.

Note that : the shortest chord of a circle passing through a point 'M' inside the circle, is one chord whose middle point is M.

$x^2+y^2+2gx+2fy+c=0$



$T=S_1$

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=$
$x_1^2+y_1^2+2gx_1+2fy_1+c.$

Problems

(i) Find the equation of the chord of $x^2 + y^2 - 6x + 10 - a = 0$ which is bisected at $(-2, 4)$.

(ii) Find the locus of mid point of chord of $x^2 + y^2 + 2gx + 2fy + c = 0$ that pass through the origin.

$x^2+y^2-6x+10-a=0$

$T = S_1$

$x(-2) + y(4) - 3(x-2) + \cancel{10-a} = 4+16+12+\cancel{10-a}$

$-2x+4y-3x+6 = 32$

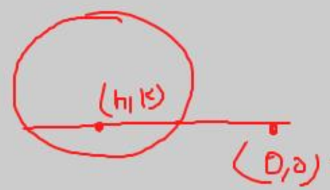
$-5x+4y=26$

$*\ \boxed{5x-4y+26=0}$

Problems

(i) Find the equation of the chord of $x^2 + y^2 - 6x + 10 - a = 0$ which is bisected at $(-2, 4)$.

(ii) Find the locus of mid point of chord of $x^2 + y^2 + 2gx + 2fy + c = 0$ that pass through the origin.



$T = S_1$

$x \times h + y \times k + g(x+h) + f(y+k) + c = h^2 + k^2 + 2gh + 2fk + c$

$x = 0,\ y = 0$

$gh + fk + \cancel{c} = h^2 + k^2 + 2gh + 2fk + \cancel{c}$

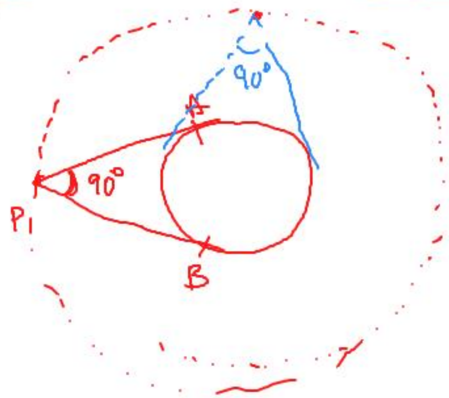
$h^2 + k^2 + gh + fk = 0$

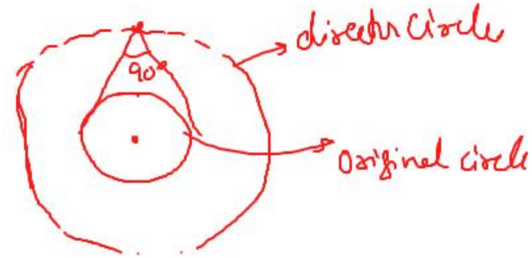
$h \to x,\ k \to y$

$\boxed{x^2 + y^2 + gx + fy = 0}$

DIRECTOR CIRCLE:

The locus of the point of intersection of two perpendicular tangents is called the **DIRECTOR CIRCLE** of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.





DIRECTOR CIRCLE:

The locus of the point of intersection of two perpendicular tangents is called the **DIRECTOR CIRCLE** of the given circle. The director circle of a circle is the concentric circle having radius equal to $\sqrt{2}$ times the original circle.

radius of original $= a$

DC $= \sqrt{2}a$

$S \equiv x^2+y^2=a^2$

Director circle $\equiv (x-0)^2+(y-0)^2=(\sqrt{2}a)^2$

$\boxed{x^2+y^2=2a^2}$

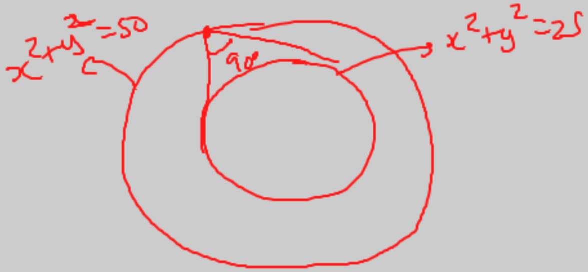
Problems

D.C

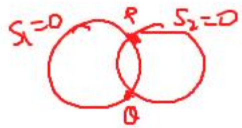
If two tangents are drawn from a point on the circle $x^2 + y^2 = 50$ to the circle $x^2 + y^2 = 25$, then find the angle between the tangents.

$x^2+y^2=a^2$

$x^2+y^2=2a^2$



$L_2=0 \quad L_1=0 \qquad L_1+\lambda L_2=0$



$\boxed{S_1+\lambda S_2=0}$

A FAMILY OF CIRCLES :

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K\,S_2 = 0 \qquad (K \neq -1)$. $\boxed{S_1 - S_2 = 0}$ not a circle.

$S_1+\lambda S_2=0$

(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.

$\boxed{S+\lambda L=0}$

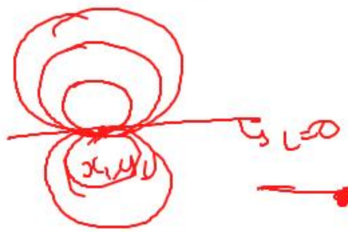
(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$\boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)} + K\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

$S+\lambda L=0$

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K\,[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.

d)

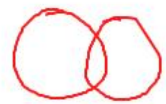


$(x_1,y_1) \quad (x_2,y_2) \quad P(x,y)$

eqn of line.

$$\frac{1}{2}\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\boxed{(x-x_1)^2+(y-y_1)^2+\lambda(L)=0}$$

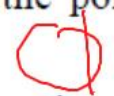


$S_1 + \lambda S_2 = 0$

A FAMILY OF CIRCLES :

(a) The equation of the family of circles passing through the points of intersection of two circles $S_1 = 0$ & $S_2 = 0$ is : $S_1 + K S_2 = 0$ $(K \neq -1)$.

(b) The equation of the family of circles passing through the point of intersection of a circle $S = 0$ & a line $L = 0$ is given by $S + KL = 0$.



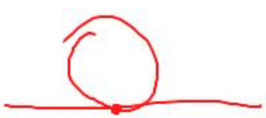
$S + \lambda L = 0$

(c) The equation of a family of circles passing through two given points (x_1, y_1) & (x_2, y_2) can be written in the form :

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + K \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$ where K is a parameter.

$S + \lambda L = 0$

(d) The equation of a family of circles touching a fixed line $y - y_1 = m(x - x_1)$ at the fixed point (x_1, y_1) is $(x - x_1)^2 + (y - y_1)^2 + K[y - y_1 - m(x - x_1)] = 0$, where K is a parameter.



P.o.i of a circle (point) & a line

$(x-x_1)^2+(y-y_1)^2+ \lambda(y-y_1-m(x-x_1)) = 0$

Problems

$S_1 = 0$

The equation of the circle through the points of intersection of $x^2 + y^2 - 1 = 0$, $x^2 + y^2 - 2x - 4y + 1 = 0$ and touching the line $x + 2y = 0$, is -

$S_2 = 0$

(A) $x^2 + y^2 + x + 2y = 0$

(B) $x^2 + y^2 - x + 20 = 0$

(C) $x^2 + y^2 - x - 2y = 0$

(D) $2(x^2 + y^2) - x - 2y = 0$

$S_1 + \lambda S_2 = 0$

$x^2 + y^2 - 1 + \lambda(x^2 + y^2 - 2x - 4y + 1) = 0$

$x^2(1+\lambda) + y^2(1+\lambda) + x(-2\lambda) - 4\lambda y + \lambda - 1 = 0$

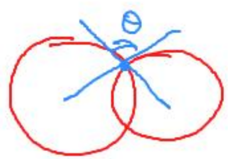
$\hookrightarrow$ touches $x + 2y = 0$

Condition of tangency

$\boxed{p = r}$

ORTHOGONALITY OF TWO CIRCLES :

Two circles $S_1=0$ & $S_2=0$ are said to be orthogonal or said to intersect orthogonally if the tangents at their point of intersection include a right angle. The condition for two circles to be orthogonal is : $2g_1g_2+2f_1f_2=c_1+c_2$.



Angle of intersection.

If angle of intersection is 90°
(orthogonal circles)

$x^2+y^2+2g_1x+2f_1y+c_1=0$

$x^2+y^2+2g_2x+2f_2y+c_2=0$

$$\boxed{2g_1g_2+2f_1f_2=c_1+c_2}$$

COMMON TANGENTS TO TWO CIRCLES :

(i) Where the two circles neither intersect nor touch each other , there are FOUR common tangents, two of them are transverse & the others are direct common tangents.

(ii) When they intersect there are two common tangents, both of them being direct.

(iii) When they touch each other :

(a) **EXTERNALLY :** there are three common tangents, two direct and one is the tangent at the point of contact .

(b) **INTERNALLY :** only one common tangent possible at their point of contact.

(iv) Length of an external common tangent & internal common tangent to the two circles is given by:

$$L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2} \quad \& \quad L_{int} = \sqrt{d^2 - (r_1 + r_2)^2} \ .$$

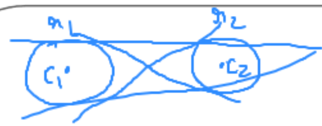
Where d = distance between the centres of the two circles . r_1 & r_2 are the radii of the two circles.

Problems

The number of common tangent(s) to the circles $x^2 + y^2 + 2x + 8y - 23 = 0$ and $x^2 + y^2 - 4x - 10y + 19 = 0$ is

(A) 1 (B) 2 (C) 3 (D) 4

(1)

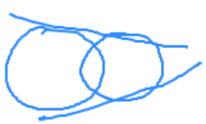


$C_1C_2 > r_1 + r_2$
(4 tangents)

(2)

$C_1C_2 = r_1 + r_2$
(3 C.T)

(3)



$r_1 - r_2 < C_1C_2 < r_1 + r_2$
2 C.T

(4)

$C_1C_2 = r_1 - r_2$
1 C.T.

(5)

$C_1C_2 < r_1 - r_2$
0 C.T

(6)

$C_1C_2 = 0$
0 C.T

Problems

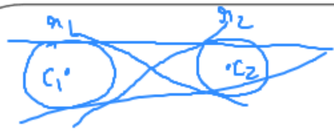
$C_2 = (2,5)$ $r_2 = \sqrt{4+25-19} = \sqrt{10}$

$C_1(-1,-4)$ $r_1 = \sqrt{1+16+23} = \sqrt{40} = 2\sqrt{10}$

The number of common tangent(s) to the circles $x^2 + y^2 + 2x + 8y - 23 = 0$ and $x^2 + y^2 - 4x - 10y + 19 = 0$ is

(A) 1 (B) 2 (C) 3 (D) 4

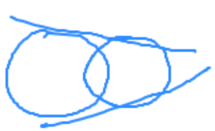
$C_1C_2 = \sqrt{3^2+9^2} = \sqrt{90} = 3\sqrt{10}$ $r_1+r_2 = \sqrt{10}+2\sqrt{10} = 3\sqrt{10}$

1 

$C_1C_2 > r_1+r_2$
(4 tangents)

2 

$C_1C_2 = r_1+r_2$
(3 C.T)

3 

$r_1-r_2 < C_1C_2 < r_1+r_2$
2 C.T

4 

$C_1C_2 = r_1-r_2$
1 C.T.

5 

$C_1C_2 < r_1-r_2$
0 C.T

6 

$C_1C_2 = 0$
0 C.T