Parabola

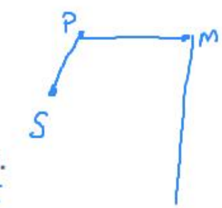
Vishal Garg (B.Tech, IIT Bombay)

Concepts

1. CONIC SECTIONS:

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

- The fixed point is called the **FOCUS.**
- The fixed straight line is called the **DIRECTRIX.**
- The constant ratio is called the **ECCENTRICITY** denoted by e.
- The line passing through the focus & perpendicular to the directrix is called the **AXIS**.
- A point of intersection of a conic with its axis is called a **VERTEX**.



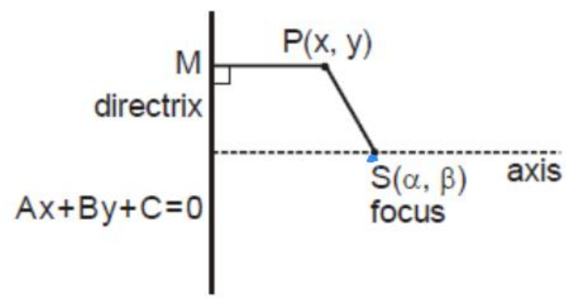
$\frac{PS}{PM}$ = constant

Conic Sections

(Parabola, Ellipse, Hyp)

*

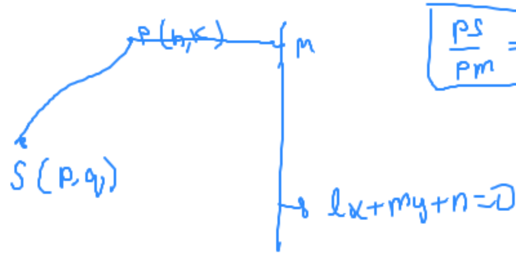
$$\boxed{\frac{PS}{PM} = e}$$



GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY:

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is:

$$\boxed{(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2} \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$



$$\boxed{\frac{PS}{PM} = e}$$

$$\sqrt{(h-p)^2 + (k-q)^2} = e\left|\frac{lh + mk + n}{\sqrt{l^2+m^2}}\right|$$

$e = e$

Problems

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Length of the latus rectum of the parabola $25\,[(x-2)^2 + (y-3)^2] = (3x - 4y + 7)^2$ is

(A) 4 (B) 2 (C) 1/5 (D) 2/5

$\frac{SP}{PM} = e$ For parabola, $e = 1$

$\boxed{SP = PM}$

$(x-2)^2 + (y-3)^2 = \left(\frac{3x-4y+7}{5}\right)^2$

** $\sqrt{(x-2)^2 + (y-3)^2} = \left|\frac{3x-4y+7}{5}\right|$

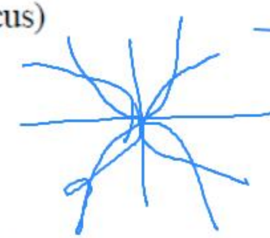
Focus $(2,3)$ Directrix: $3x-4y+7=0$ $e=1$

PARABOLA : DEFINITION : (e=1)

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

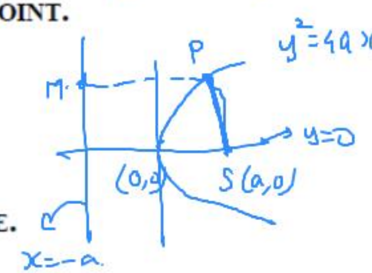
Standard equation of a parabola is $y^2 = 4ax$. For this parabola:

(i) Vertex is (0, 0)   (ii) focus is (a, 0)   (iii) Axis is $y = 0$ (iv) Directrix is $x + a = 0$



FOCAL DISTANCE :

The distance of a point on the parabola from the focus is called the **FOCAL DISTANCE OF THE POINT.**



FOCAL CHORD :

A chord of the parabola, which passes through the focus is called a **FOCAL CHORD.**

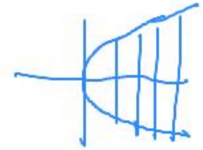
DOUBLE ORDINATE :

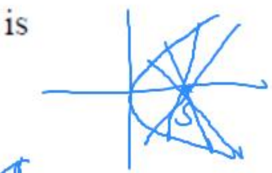
A chord of the parabola perpendicular to the axis of the symmetry is called a **DOUBLE ORDINATE.**

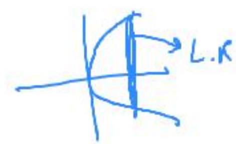
LATUS RECTUM :

A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the **LATUS RECTUM.** For $y^2 = 4ax$.

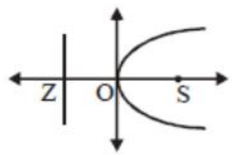
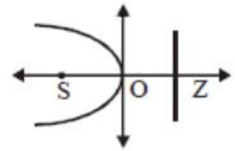
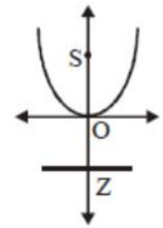
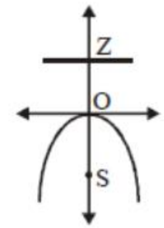
- Length of the latus rectum = 4a.
- ends of the latus rectum are L(a, 2a) & L' (a, – 2a).







Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$

Equation of parabola	**$y^2 = 4ax$**	**$y^2 = -4ax$**	**$x^2 = 4ay$**	**$x^2 = -4ay$**
(a) Graphs				
(b) Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
(c) Focus	$S(a, 0)$	$S(-a, 0)$	$S(0, a)$	$S(0, -a)$
(d) Equation of directrix	$x + a = 0$	$x - a = 0$	$y + a = 0$	$y - a = 0$
(e) Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
(f) Vertex	$O(0, 0)$	$O(0, 0)$	$O(0, 0)$	$O(0, 0)$
(g) Extremities of latusrectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
(h) Length of latusrectum	$4a$	$4a$	$4a$	$4a$
(i) Equation of tangent at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
(j) Parametric coordinates of any point on parabola	$\boxed{P(at^2, 2at)}$	$P(-at^2, 2at)$	$P(2at, at^2)$	$P(2at, -at^2)$

$y^2 = 4ax$

$x = at^2, \; y = 2at$ ✓

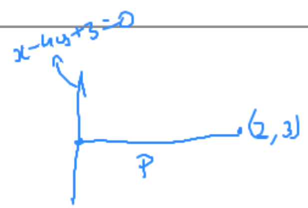
$y^2 = -4ax$

$x = -at^2, \; y = 2at$

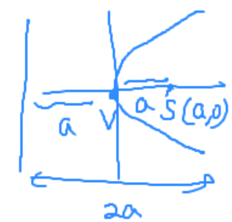
Problems

The length of latus rectum of a parabola, whose focus is (2, 3) and directrix is the line $x - 4y + 3 = 0$ is -

(A) $\frac{7}{\sqrt{17}}$ (B) $\frac{14}{\sqrt{21}}$ (C) $\frac{7}{\sqrt{21}}$ (D) $\frac{14}{\sqrt{17}}$



$$\left|\frac{2-12+3}{\sqrt{1+16}}\right| = \frac{7}{\sqrt{17}}$$



LR = 4a.

L.R = twice of dist b/w focus & directrix.

$S_1 > 0$ (outside)
$S_1 = 0$ (on)
$S_1 < 0$ (inside)

POSITION OF A POINT RELATIVE TO A PARABOLA :

The point $(x_1\ y_1)$ lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

Hence in short, equation of parabola $S(x, y) = \boxed{y^2 - 4ax}$.

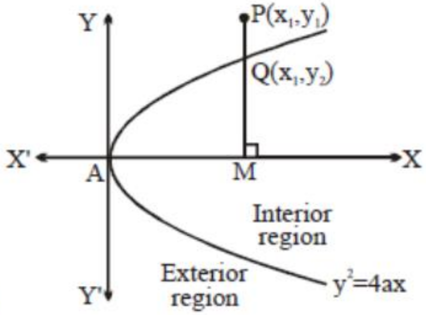
(i) If $S(x_1, y_1) > 0$ then $P(x_1, y_1)$ lie outside the parabola.

(ii) If $S(x_1, y_1) < 0$ then $P(x_1, y_1)$ lie inside the parabola.

(iii) If $S(x_1, y_1) = 0$ then $P(x_1, y_1)$ lie on the parabola.

This result holds true for circle, parabola and ellipse.

(Not hyperbola)



Problems

Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

No values of α for which point lies inside

$S = y^2 - 4x$ $\quad\quad P(\alpha-1, \alpha)$

$S_1 = \alpha^2 - 4(\alpha - 1)$

Inside : $S_1 < 0$

$\alpha^2 - 4\alpha + 4 < 0$

$(\alpha - 2)^2 < 0$

$\alpha \in \phi$

CHORD JOINING TWO POINTS :

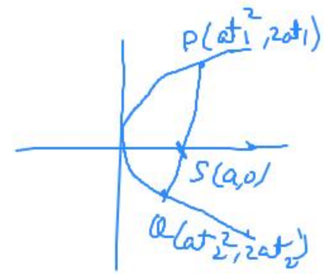
The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is

$$\boxed{y(t_1 + t_2) = 2x + 2at_1t_2}$$

Note :

(i) If PQ is focal chord then $\boxed{t_1t_2 = -1}$.

(ii) Extremities of focal chord can be taken as $(at^2,\ 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$



LINE & A PARABOLA :

(a) The line $y = mx + c$ meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as $\boxed{a >=< cm}$ $\Rightarrow$ condition of tangency is, $\boxed{c = \frac{a}{m}}$.

Note : Line $y = mx + c$ will be tangent to parabola $x^2 = 4ay$ if $c = -am^2$.

$\boxed{x+y+1=0}$ → tangent

$x = -y-1$

$y^2 = 4x$

$y^2 = 4(-y-1)$

$\boxed{y^2 + 4y + 4 = 0}$

$D = 16 - 16 = 0$

$D>0 \quad a>cm$

$D=0 \quad a=cm \Rightarrow \boxed{c = \frac{a}{m}}$

$D<0 \quad a<cm$

→ 2 distinct points Secant $D>0$

→ 2 coincident points Tangent $D=0$

→ img points Outside $D<0$

$\boxed{y = mx + \frac{a}{m}}$ → eqn of tangent to $y^2 = 4ax$.

Problems

If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct points then set of values of λ is -

(A) $(3, \infty)$ (B) $(-\infty, 1/3)$ (C) $(1/3, 3)$ (D) none of these

$y = mx + c$ $y^2 = 4ax$ For 2 point of intersection

$y = 3x + \lambda$ $y^2 = 4x$ $D > 0$ or $\boxed{a > cm}$

M-1 $c = \lambda$, $m = 3$, $a = 1$

$1 > 3\lambda \Rightarrow \lambda < \frac{1}{3}$

M-2

$y = 3x + \lambda$ $y^2 = 4x$

$x = \frac{y - \lambda}{3} \Rightarrow y^2 = \frac{4}{3}(y - \lambda)$

$\boxed{3y^2 - 4y + 4\lambda = 0}$ $D > 0$

$16 - 48\lambda > 0$

$48\lambda < 16$

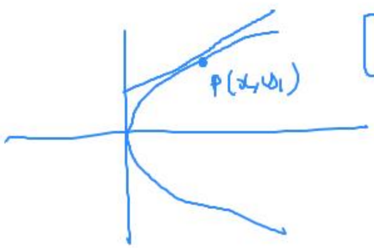
$\lambda < \frac{1}{3}$

TANGENTS TO THE PARABOLA $y^2 = 4ax$:

(i) $T=0$ $yy_1 = 2a(x + x_1)$ at the point (x_1, y_1) ; **(ii)** $y = mx + \frac{a}{m}$ $(m \neq 0)$ at $\boxed{\left(\frac{a}{m^2}, \frac{2a}{m}\right)}$

(iii) $ty = x + at^2$ at $(at^2, 2at)$.

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[\,at_1 t_2,\ a(t_1 + t_2)\,]$.

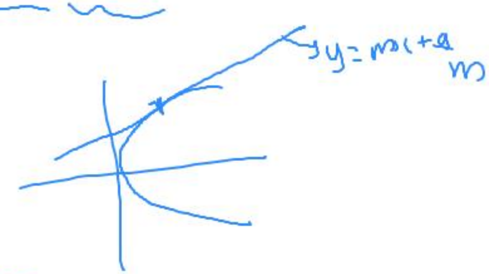


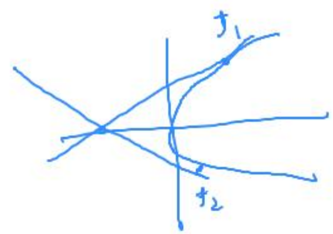
$\boxed{T=0}$

$y^2 = 4ax$

$\boxed{yy_1 = 2a(x+x_1)}$

$x_1 = at^2,\ y_1 = 2at$



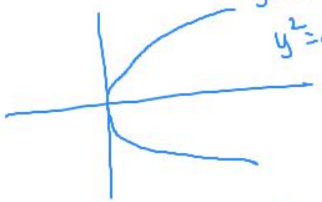


Problems

A tangent to the parabola $y^2 = 8x$ makes an angle of 45 with the straight line $y = 3x + 5$. Find its equation and its point of contact. $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$y = 3x + 5$ $\quad 45^\circ \quad$ Slope of tangent $= m$

$m_1 = 3 \quad \theta = 45^\circ$



$y^2 = 8x$

$y^2 = 4ax$

$a = 2$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$1 = \left|\frac{3 - m}{1 + 3m}\right|$

Slope of tangent: $\frac{1}{2}$ & 2

$\frac{3-m}{1+3m} = \pm 1 \Rightarrow 3 - m = 1 + 3m \quad \Big| \quad 3 - m = -1 - 3m$

$4m = 2 \quad\quad 2m = -4$

$m = 1/2 \quad\quad m = -2$

eqn of tangent: $y = mx + \frac{a}{m}$

$y = \frac{1}{2}x + \frac{2}{1/2}$

$\boxed{y = \frac{1}{2}x + 4}$

$y = 2x + \frac{2}{2}$

$\boxed{y = 2x + 1}$

NORMALS TO THE PARABOLA $y^2 = 4ax$:

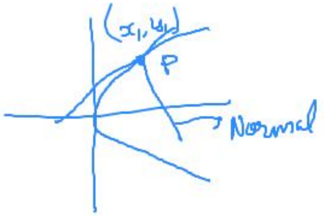
(i) $y - y_1 = -\frac{y_1}{2a}(x - x_1)$ at (x_1, y_1) ; **(ii)** $y = mx - 2am - am^3$ at $(am^2, -2am)$

(iii) $y + tx = 2at + at^3$ at $(at^2, 2at)$.

Note : Point of intersection of normals at t_1 & t_2 are, $a(t_1^2 + t_2^2 + t_1t_2 + 2)$; $-at_1t_2(t_1 + t_2)$.

$y^2 = 4ax$ $\quad$ $y = mx + \frac{a}{m}$ (always tangent)

$\rightarrow$ $y = mx - 2am - am^3$ (always normal)

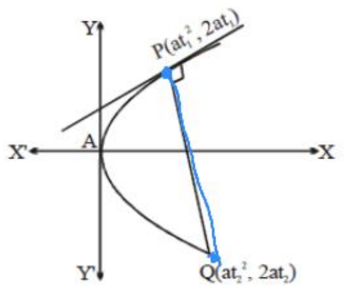


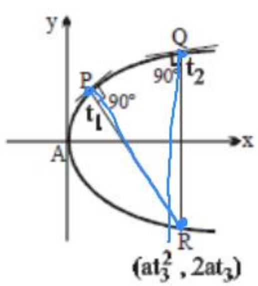
THREE VERY IMPORTANT RESULTS :

(a) If t_1 & t_2 are the ends of a focal chord of the parabola $y^2 = 4ax$ then $\boxed{t_1 t_2 = -1}$. Hence the co-ordinates at the extremities of a focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

(b) If the normals to the parabola $y^2 = 4ax$ at the point t_1 meets the parabola again at the point t_2, then $\boxed{t_2 = -\left(t_1 + \frac{2}{t_1}\right)}$.

(c) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $\boxed{t_1 t_2 = 2 \,;\, t_3 = -(t_1 + t_2)}$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.





Problems

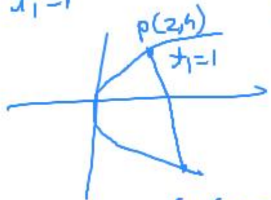
Normal to the parabola $y^2 = 8x$ at the point P (2, 4) meets the parabola again at the point Q. If C is the centre of the circle described on PQ as diameter then the coordinates of the image of the point C in the line $y = x$ are

(A) (– 4, 10) (B) (– 3, 8) (C) (4, – 10) (D) (– 3, 10)

$2 = at^2,\ 4 = 2at$

$t_1 = 1$

$y^2 = 8x \quad a = 2$



$P(at_1^2, 2at_1) \quad Q(at_2^2, 2at_2)$

$t_2 = -t_1 - \frac{2}{t_1}$

$t_2 = -1 - 2$

$Q\ (t_2 = -3)$

$Q(at_2^2, 2at_2)$

$2(9), 2\times2(-3)$

$(18, -12)$

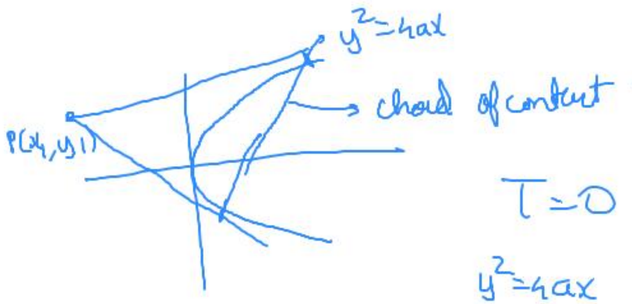
$\boxed{P(2,4),\ Q(18,-12)}$

C is centre i.e midpoint of PQ $\equiv (10, -4)$

image in $(y = x)$ is $(-4, 10)$

The equation to the pair of tangents which can be drawn from any point (x_1, y_1) to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$ where :

$S \equiv y^2 - 4ax$; $S_1 = y_1^2 - 4ax_1$; $T \equiv y y_1 - 2a(x + x_1)$.



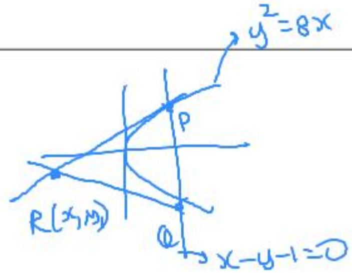
$T = 0$

$y^2 = 4ax$

$yy_1 = 2a(x+x_1) \longrightarrow$ C.O.C

Problems

If the line x – y – 1 = 0 intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.



What is PQ w.r.t R (C.O.C)

$T = 0$

$yy_1 = 4(x + x_1)$ eqn of PQ.

$$\left.\begin{array}{ll} 4x - y_1 y + 4x_1 = 0 & PQ \\ x - y - 1 = 0 & PQ \end{array}\right\} \text{Same}$$

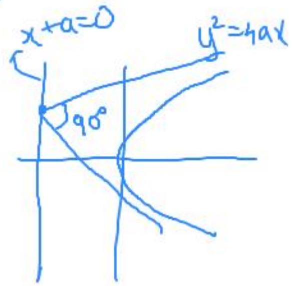
$$\frac{4}{1} = \frac{y_1}{1} = \frac{4x_1}{-1} \Rightarrow x_1 = -1,\ y_1 = 4.$$

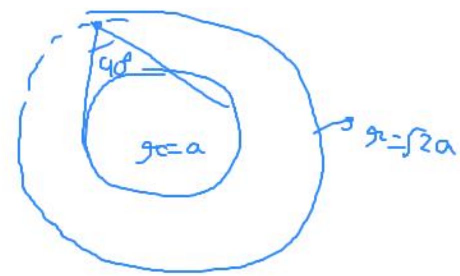
$(-1, 4)$

does not represent shape.

DIRECTOR CIRCLE :

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the **DIRECTOR CIRCLE.** It's equation is $x + a = 0$ which is parabola's own directrix.





Directrix is the Director circle for parabola.

Problems

this point lies on directrix.

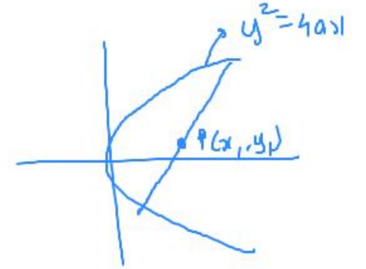
The angle between the tangents drawn from a point (–a, 2a) to $y^2 = 4ax$ is -

(A) $\pi/4$ (B) $\pi/2$ (C) $\pi/3$ (D) $\pi/6$

CHORD WITH A GIVEN MIDDLE POINT :

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$. This reduced to $\boxed{T = S_1}$

where $T \equiv y y_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

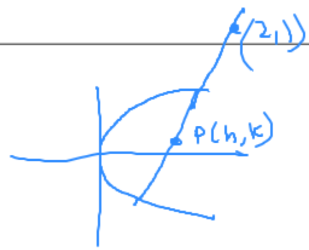


$\boxed{yy_1 - 2a(x+x_1) = y_1^2 - 4ax_1}$

is eqn of chord whose middle point is (x_1, y_1)

Problems

Find the locus of middle point of the chord of the parabola $y^2 = 4ax$ which pass through a given (p, q). → (2,1)



eqn of chord = $T = S_1$

$yk - 2a(x+h) = k^2 - 4ah$ → passes through (2,1)

$k - 2a(2+h) = k^2 - 4ah$

$k - 4a - 2ah = k^2 - 4ah$

$k^2 - 2ah - k + 4a = 0$

$\boxed{y^2 - 2ax - y + 4a = 0}$ → Locus