PHYSICS

NEET and JEE Main 2020 : 45 Days Crash Course

Simple Harmonic Motion

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Periodic and Oscillatory Motion

Periodic Motion

Any motion which repeats itself after regular interval of time along a definite path is called periodic motion or harmonic motion.

The constant interval of time after which the motion is repeated is called time period.

Examples : (i) Motion of planets around the sun.
(ii) Motion of the pendulum of wall clock.

Oscillatory Motion

The motion of body is said to be oscillatory or vibratory motion if it moves back and forth (to and fro) about a fixed point after regular interval of time.

The fixed point about which the body oscillates is called mean position or equilibrium position.

Examples : (i) Vibration of the wire of 'Sitar'.
(ii) Oscillation of the mass suspended from spring.

Note : Every oscillatory motion is periodic but every periodic motion is not oscillatory.

Simple harmonic motion (S.H.M.)

Necessary Condition to execute S.H.M.

(a) Motion of particle should be oscillatory.

(b) Total mechanical energy of particle should be conserved (Kinetic energy + Potential energy = constant)

(c) **In linear S.H.M.**

The restoring force (or acceleration) acting on the particle should always be proportional to the displacement of the particle and directed towards the equilibrium position

$\therefore$ $\boxed{F \propto -x}$ or $\boxed{a \propto -x}$

Negative sign shows that direction of force and acceleration is towards equilibrium position and x is displacement of particle from equilibrium position.

(d) **In angular S.H.M.**

The restoring torque (or angular acceleration) acting on the particle should always be proportional to the angular displacement of the particle and directed towards the equilibrium position

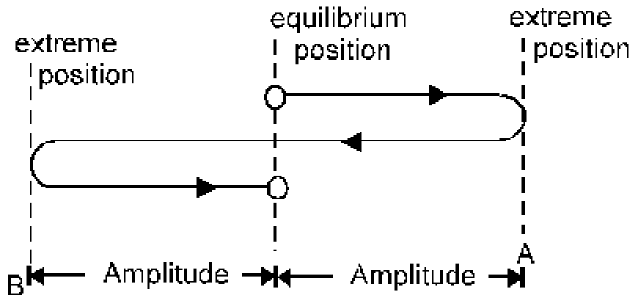
$\therefore$ $\boxed{\tau \propto -\theta}$ or $\boxed{\alpha \propto -\theta}$

Comparison between linear and angular S.H.M.

Linear S.H.M.	Angular S.H.M.
$F \propto x$	$\tau \propto \theta$
$F \quad kx$ $ma = -kx$	$\tau \quad C\theta$
Where k is the restoring force constant	Where C is the restoring torque constant
$a \quad -\frac{k}{m}x = -\omega^2 x$	$\alpha \quad -\frac{C}{I}\theta$
$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$	$\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0$
It is known as differential equation of linear S.H.M.	It is known as differential equation of angular S.H.M.
$\boxed{x \quad A \sin\omega t}$ $A \sin(\omega t + \phi)$	$\theta \quad \theta_0 \sin \omega t$
$\boxed{a = -\omega^2 x}$	$\alpha = -\omega^2\theta$
where ω is the angular frequency	
$\omega^2 = \frac{k}{m}$	$\omega^2 = \frac{C}{I}$
$\boxed{\omega = \sqrt{\frac{k}{m}}} = \frac{2\pi}{T} = 2\pi n$	$\omega = \sqrt{\frac{C}{I}} = \frac{2\pi}{T} = 2\pi n$
where T is time period and n is frequency	
$T = 2\pi\sqrt{\frac{m}{k}} = \frac{2\pi}{\omega}$	$T = 2\pi\sqrt{\frac{I}{C}}$
$n \quad \frac{1}{2\pi}\sqrt{\frac{k}{m}}$	$n \quad \frac{1}{2\pi}\sqrt{\frac{C}{I}}$
This concept is valid for all types of linear S.H.M.	This concept is valid for all types of angular S.H.M.

Some basic terms in S.H.M.

(a) Displacement - It is defined as the distance of the particle from the mean position at that instant. Displacement in SHM at time t is given by $\boxed{x = A \sin(\omega t + \phi)}$



(b) Amplitude - It is the maximum value of displacement of the particle from its equilibrium position.

Amplitude = $\frac{1}{2}$ [distance between extreme points or positions]

It depends on energy of the system.

(c) Angular Frequency (ω) : $\omega = \frac{2\pi}{T} = 2\pi f$ and its unit is rad/sec.

(d) Frequency (f) : Number of oscillations completed in unit time interval is called frequency of oscillations, $f = \frac{1}{T} = \frac{\omega}{2\pi}$, its units is sec^{-1} or Hz.

(e) Time period (T) : Smallest time interval after which the oscillatory motion gets repeated is called time period, $\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}$

Some basic terms in S.H.M.

(f) Phase : The physical quantity which represents the state of motion of particle (eg. its position and direction of motion at any instant).
The argument ($\omega t + \phi$) of sinusoidal function is called instantaneous phase of the motion.

(g) Phase constant (ϕ) : Constant ϕ in equation of SHM is called phase constant or initial phase.
It depends on initial position and direction of velocity.

(h) Velocity(v) : Velocity at an instant is the rate of change of particle's position w.r.t time at that instant.
Let the displacement from mean position is given by, $x = A \sin(\omega t + \phi)$

Velocity, $v = \dfrac{dx}{dt} = \dfrac{d}{dt}[A\sin(\omega t + \phi)]$

$v = A\omega \cos(\omega t + \phi)$ or, $\boxed{v = \omega\sqrt{A^2 - x^2}}$ speed

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$$

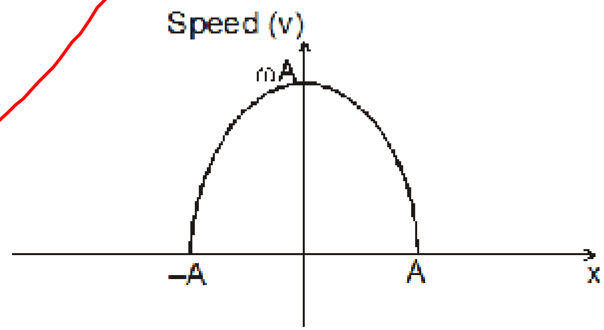
At mean position ($x = 0$), velocity is maximum. $v_{max} = \omega A$
At extreme position ($x = A$), velocity is minimum. v_{min} = zero

GRAPH OF SPEED (v) VS DISPLACEMENT (x):

$$v = \omega\sqrt{A^2 - x^2} \qquad v^2 = \omega^2(A^2 - x^2)$$

$$v^2 + \omega^2 x^2 = \omega^2 A^2 \qquad \frac{v^2}{\omega^2 A^2} + \frac{x^2}{A^2} = 1$$

GRAPH WOULD BE AN ELLIPSE



Some basic terms in S.H.M.

(i) Acceleration : Acceleration at an instant is the rate of change of particle's velocity w.r.t. time at that instant.

$$\text{Acceleration, } a = \frac{dv}{dt} = \frac{d}{dt}[A\omega \cos(\omega t + \phi)]$$

$$a = -\omega^2 A \sin(\omega t + \phi)$$

$$a = -\omega^2 x$$

Note • Negative sign shows that acceleration is always directed towards the mean position.
At mean position (x = 0), acceleration is minimum.

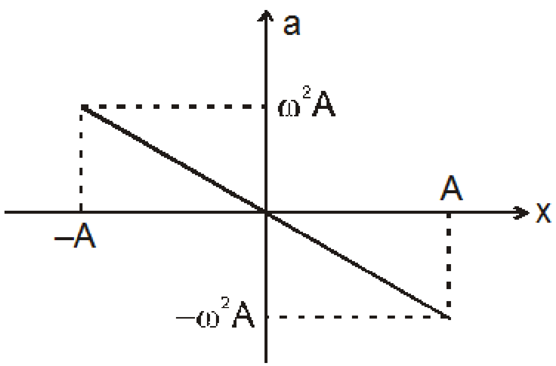
$$a_{min} = \text{zero}$$

At extreme position (x = A), acceleration is maximum.

$$a_{max} = \omega^2 A$$

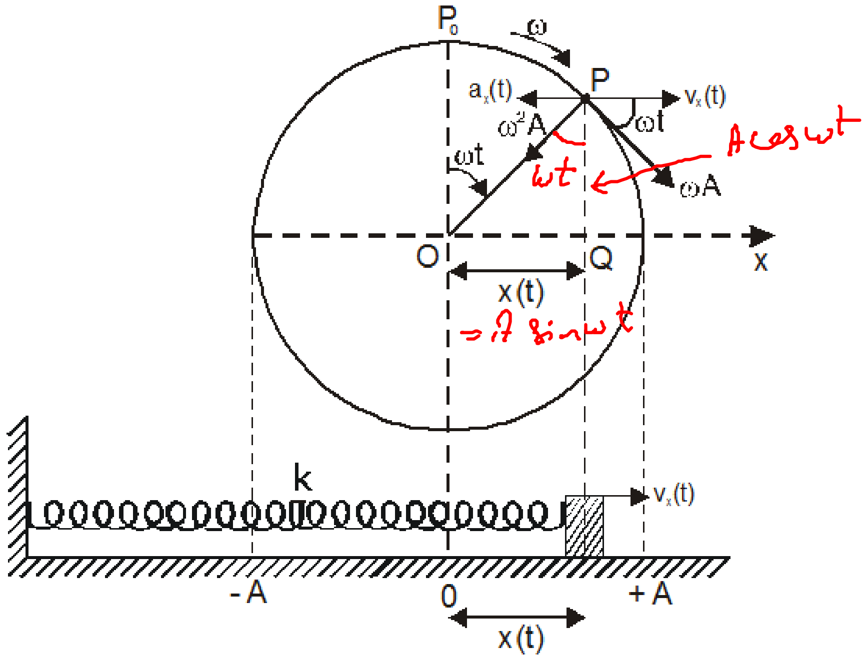
GRAPH OF ACCELERATION (A) VS DISPLACEMENT (x)

$$a = -\omega^2 x$$



SHM as a projection of uniform circular motion

Consider a particle moving on a circle of radius A with a constant angular speed ω as shown in figure.



Suppose the particle is on the top of the circle (Y-axis) at t = 0. The radius OP makes an angle $\theta = \omega t$ with the Y-axis at time t. Drop a perpendicular PQ on X-axis. The components of position vector on the X-axis

$$x(t) = A \sin \omega t$$

Above equations show that the foot of perpendicular Q executes a simple harmonic motion on the X-axis. The amplitude is A and angular frequency is ω. Similarly the foot of perpendicular on Y-axis will also execute SHM of amplitude A and angular frequency ω [$y(t) = A \cos \omega t$].

Graphical representation of displacement, velocity & acceleration in SHM

sin(90+θ) = cos θ

Displacement,

$x = A \sin \omega t$

v leads x by phase angle of π/2

Velocity,

$v = A\omega \cos \omega t = A\omega \sin (\omega t + \frac{\pi}{2})$ → 90°

or $v = \omega \sqrt{A^2 - x^2}$

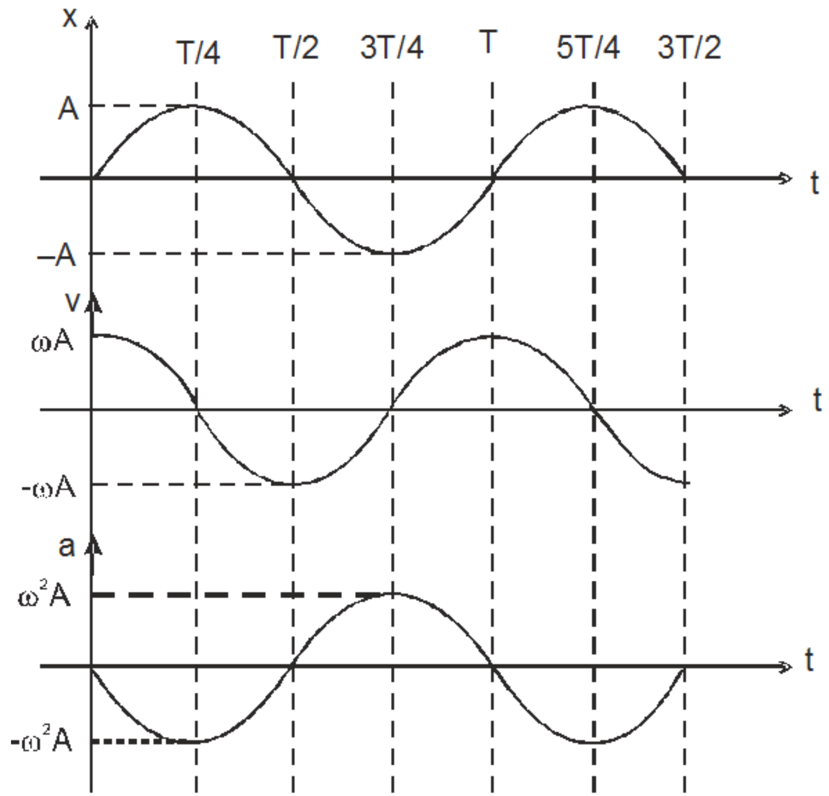
v lags behind a by π/2

a leads v by phase angle of π/2

Acceleration,

$a = -\omega^2 A \sin\omega t = \omega^2 A \sin (\omega t + \pi)$ ← 180°

or $a = -\omega^2 x$



Energy of SHM

Kinetic Energy (KE)

$\omega = \sqrt{k/m}$

$$\frac{1}{2}mv^2 = \frac{1}{2}m\omega^2(A^2 - x^2) = \boxed{\frac{1}{2}k(A^2 - x^2)} \text{ (as a function of x)}$$

$$\frac{1}{2}mA^2\omega^2\cos^2(\omega t + \theta) = \frac{1}{2}KA^2\cos^2(\omega t + \theta) \text{ (as a function of t)}$$

$$KE_{max} = \frac{1}{2}kA^2 \quad ; \quad \langle KE \rangle_{0-T} = \frac{1}{4}kA^2 \quad ;$$

Frequency of KE = 2 × (frequency of SHM)

Total Mechanical Energy (TME)

Total mechanical energy = KE + PE

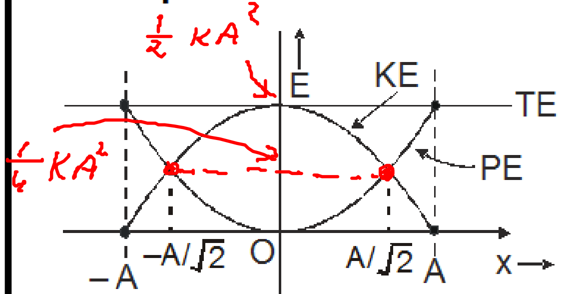
$$= \frac{1}{2}k(A^2 - x^2) + \frac{1}{2}Kx^2 = \frac{1}{2}KA^2$$

Hence total mechanical energy is constant in SHM.

Potential Energy (PE)

$$\frac{1}{2}Kx^2 \text{ (as a function of x)} = \frac{1}{2}kA^2\sin^2(\omega t + \theta) \text{ (as a function of time)}$$

Graphical Variation of energy of particle in SHM



Steps to show SHM

1. Find the equilibrium position.
2. Displace the particle by small amount (x) from the equilibrium position.
3. Find resultant force on the particle in this new position.
4. If resultant force is proportional to x and is directed towards equilibrium position, then particle will perform SHM.

$$F = -Kx$$

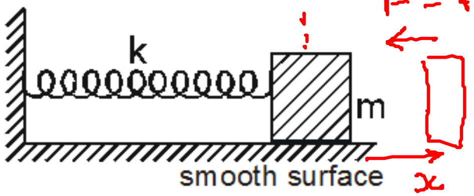
5. Time period will be

$$T = 2\pi\sqrt{\frac{m}{K}}$$

$F = -Kx$

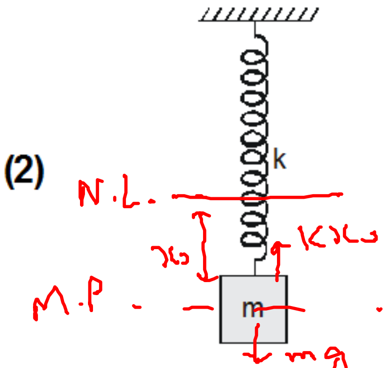
Combination of Springs

(1)



$$T = 2\pi\sqrt{\frac{m}{k}}$$

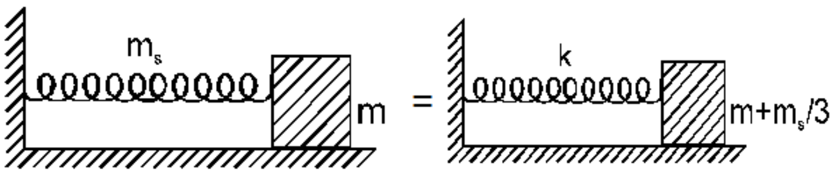
(2)



$$T = 2\pi\sqrt{\frac{m}{k}}$$

$x_0 = \frac{mg}{K}$

(3) If spring has mass m_s then



$$T = 2\pi\sqrt{\frac{m+\frac{m_s}{3}}{k}}$$

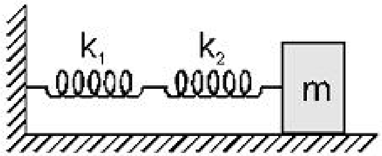
Constant force does not affect T.P. or freq. of linear S.H.M. They only changes the M.P.

Combination of Springs

Series Combination

Equivalent spring constant K_{eq} is given by :

$1/k_{eq} = 1/k_1 + 1/k_2 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$



$K_{eq.} = \frac{K_1 K_2}{K_1 + K_2}$

$T = Kx = \text{const}$

$x \propto \frac{1}{K}$

Note :

- In series combination, tension is same in all the springs & extension will be different. (If k is same then deformation is also same)
- In series combination , extension of springs will be reciprocal of its spring constant.
- Spring constant of spring is reciprocal of its natural length

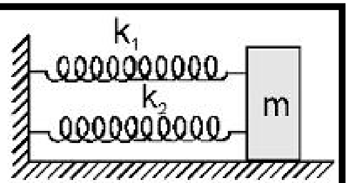
  $\because\ k \propto 1/\ell$

  $\therefore\ k_1\ell_1 = k_2\ell_2 = k_3\ell_3$

  $K_1 l_1 = K_2 l_2 = K l$
- If a spring is cut in ‘n’ pieces then spring constant of one piece will be nk.

Parallel combination

$k_{eq} = k_1 + k_2 \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{m}{k_{eq}}}$

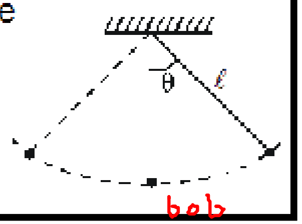


Simple Pendulum

If a heavy point mass is suspended by a weightless, inextensible and perfectly flexible string from a rigid support, then this arrangement is called a simple pendulum

Time period of a simple pendulum $T = 2\pi\sqrt{\frac{\ell}{g}}$ $\ell << R$

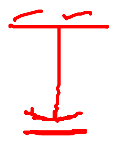
(some times we can take $g = \pi^2$ for making calculation simple)



Note :

- If angular amplitude of simple pendulum is more, then time period $T = 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{\theta_0^2}{16}\right)$ where θ_0 is in radians.
- General formula for time period of simple pendulum when ℓ is comparable to radius of Earth R.

$$T = 2\pi\sqrt{\frac{1}{g\left(\frac{1}{R} + \frac{1}{\ell}\right)}}$$ where, R = Radius of the earth



- Time period of simple pendulum of infinite length is maximum and is given by: $T = 2\pi\sqrt{\frac{R}{g}} = 84.6$ min

  (Where R is radius of earth)
- Time period of seconds pendulum is 2 sec and $\ell = 0.993$ m.
- Simple pendulum performs angular S.H.M. but due to small angular displacement, it is considered as linear S.H.M.
- If time period of clock based on simple pendulum increases then clock will be slow but if time period decrease then clock will be fast.

Time Period of Simple Pendulum in accelerating Reference Frame

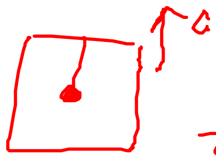
$$T = 2\pi\sqrt{\frac{\ell}{g_{eff.}}} \quad \text{where}$$

$g_{eff.}$ = Effective acceleration in accelerating reference system = $|\vec{g} - \vec{a}|$, at mean position

$\vec{a}$ = acceleration of the point of suspension w.r.t. ground.

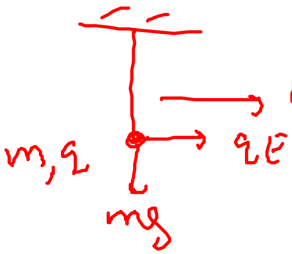
Condition for applying this formula: $|\vec{g} - \vec{a}|$ = constant

Also $g_{eff} = \dfrac{\text{Net tension in string}}{\text{mass of bob}}$ at mean position



$g_{eff} = g + a$

$T = 2\pi\sqrt{\frac{\ell}{g+a}}$



$F_{net} = \sqrt{m^2g^2 + q^2E^2}$

$g_{eff} = \frac{F_{net}}{m}$

If forces other then $m\vec{g}$ acts then

$$T = 2\pi\sqrt{\frac{\ell}{g_{eff.}}} \quad \text{where} \quad g_{eff.} = \left|\vec{g} + \frac{\vec{F}}{m}\right|$$

$\vec{F}$ = constant force acting on 'm'.

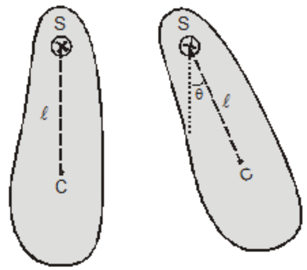
Compound Pendulum / Physical Pendulum

When a rigid body is suspended from an axis and made to oscillate about that then it is called compound pendulum.

C = Position of center of mass

S = Point of suspension

ℓ = Distance between point of suspension and center of mass (it remains constant during motion)



Time period, $T = 2\pi\sqrt{\frac{I}{mg\ell}}$ $\quad I = I_{CM} + m\ell^2$

where, I = Moment of inertia about point of suspension

$T = 2\pi\sqrt{\frac{I}{mg\ell}}$

$I = \frac{mR^2}{2} + mR^2 = \frac{3}{2}mR^2$

$\ell = R$

$T = 2\pi\sqrt{\frac{\frac{3}{2}\cancel{m}R^{\cancel{2}}}{\cancel{m}g\cancel{R}}}$

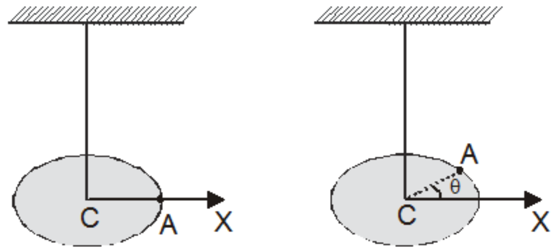
$= 2\pi\sqrt{\frac{3R}{2g}}$

Eff. length of simple pendulum

$\ell_{eff} = \frac{3R}{2}$

Torsional Pendulum

In torsional pendulum, an extended object is suspended at the centre by a light torsion wire. A torsion wire is essentially inextensible, but is free to twist about its axis. When the lower end of the wire is rotated by a slight amount, the wire applies a restoring torque causing the body to oscillate (rotate) about vertical wire, when released. The restoring torque produced is given by



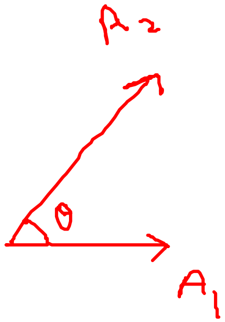
$\tau = -C\theta$ where, C = Torsional constant

or, $I\alpha = -C\theta$ where, I = Moment of inertia about the vertical axis.

or, $\alpha = -\dfrac{C}{I}\theta$ $\alpha = -\omega^2\theta$

$\therefore$ $\boxed{\text{Time Period, } T = 2\pi\sqrt{\dfrac{I}{C}}}$

Superposition of two SHM's



In same direction and of same frequency

$x_1 = A_1 \sin \omega t$

$x_2 = A_2 \sin (\omega t + \theta)$, then resultant displacement

$x = x_1 + x_2 = A_1 \sin \omega t + A_2 \sin (\omega t + \theta) = A \sin (\omega t + \phi)$

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\theta}$ & $\phi = \tan^{-1}\left[\dfrac{A_2 \sin\theta}{A_1 + A_2\cos\theta}\right]$

If $\theta = 0$, both SHM's are in phase and $A = A_1 + A_2$

If $\theta = \pi$, both SHM's are out of phase and $A = |A_1 - A_2|$

In same direction but are of different frequencies

$x_1 = A_1 \sin \omega_1 t$ $x_2 = A_2 \sin \omega_2 t$

then resultant displacement $x = x_1 + x_2 = A_1 \sin \omega_1 t + A_2 \sin \omega_2 t$ This resultant motion is not SHM.

$x = A \sin(\omega t + \phi)$

Example

For a particle performing SHM, equation of motion is given as $\frac{d^2x}{dt^2} + 4x = 0$. Find the time period.

Sol.

$$\frac{d^2x}{dt^2} = -4x$$

$\omega^2 = 4$ or $\omega = 2$

Time period; $T = \frac{2\pi}{\omega} = \pi$

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

Example

The equation of particle executing simple harmonic motion is $x = (5\text{ m})\sin\left[(\pi\, s^{-1})t + \frac{\pi}{3}\right]$. Write down the amplitude, time period and maximum speed. Also find the velocity at t = 1 s.

Sol.

Comparing with equation $x = A\sin(\omega t + \delta)$,
we see that the amplitude = 5 m,

and time period $= \frac{2\pi}{\omega} = \frac{2\pi}{\pi\, s^{-1}} = 2s$.

The maximum speed $= A\omega = 5\text{ m} \times \pi\, s^{-1} = 5\pi$ m/s.

The velocity at time $t = \frac{dx}{dt} = A\omega\cos(\omega t + \delta)$

At $t = 1$ s,

$$v = (5\text{ m})(\pi\, s^{-1})\cos\left(\pi + \frac{\pi}{3}\right) = -\frac{5\pi}{2}\text{ m/s}.$$

$x = A\sin(\omega t + \phi)$

$v = \omega\sqrt{A^2 - x^2}$

$A = 5m$

$\omega = \pi$

$T = \frac{2\pi}{\omega} = 2s$

$v_{max} = \omega A = 5\pi\ m/s$

$v = \frac{dx}{dt} = 5\pi\cos(\pi t + \pi/3)$

At $t = 1s$

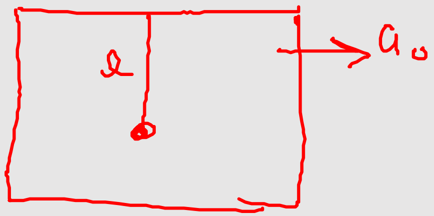
$v = 5\pi\cos(\pi + \pi/3) = -\frac{5\pi}{2}$ m/s

$= -5\pi\cos\pi/3$

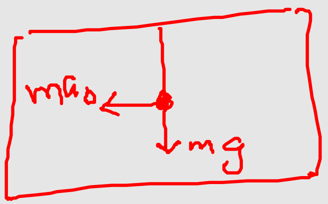
Example

A simple pendulum is suspended from the ceiling of a car accelerating uniformly on a horizontal road. If the acceleration is a_0 and the length of the pendulum is ℓ, find the time period of small oscillations about the mean position.

Sol.



w.r.t. car



$$g_{eff} = \sqrt{g^2 + a_0^2}$$

$$T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2+a_0^2}}}$$