Straight Lines

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Problems

The number of points on x-axis which are at a distance c (c < 3) from the point (2, 3) is

B

(A) 2 (B) 1 (C) infinite (D) no point

x-axis A $(x_1, 0)$

$AB = c$

$\sqrt{(x_1-2)^2+3^2} = c$

$(x_1-2)^2+3^2 = c^2$

$(x_1-2)^2 = c^2-9$

$x_1-2 = \pm\sqrt{c^2-9}$

$x_1 = 2\pm\sqrt{c^2-9}$

$c<3$

$c^2<9$

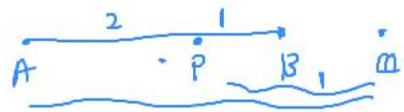
$c^2-9<0$

x_1 is imaginary

Concepts

DISTANCE FORMULA :

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$



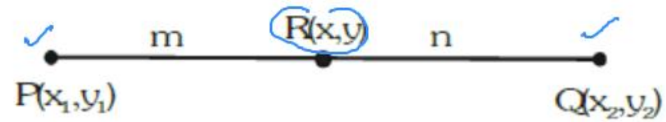
P divides AB in 2:1

SECTION FORMULA :

The co-ordinates of a point dividing a line joining the points $P(x_1,y_1)$ and $Q(x_2,y_2)$ in the ratio m:n is given by :

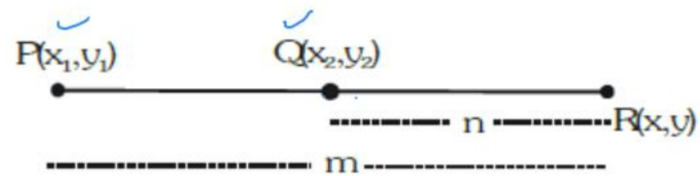
(a) For internal division : P - R - Q $\Rightarrow$ R divides line segment PQ, internally.

$$\boxed{(x,\ y) \equiv \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)}$$



(b) For external division : R - P - Q or P - Q - R $\Rightarrow$ R divides line segment PQ, externally.

$$(x,\ y) \equiv \left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

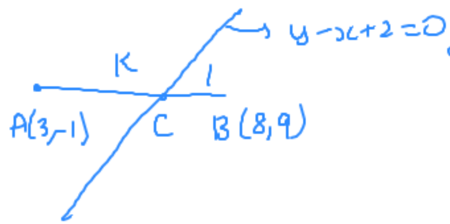


$\frac{(PR)}{(QR)} < 1 \Rightarrow$ R lies on the left of P & $\frac{(PR)}{(QR)} > 1 \Rightarrow$ R lies on the right of Q

Note : If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Problems

Determine the ratio in which $y - x + 2 = 0$ divides the line joining $(3, -1)$ and $(8, 9)$.



$$C\left(\frac{8K+3}{K+1}, \frac{9K-1}{K+1}\right)$$

$$\frac{9K-1}{K+1} - \frac{8K+3}{K+1} + 2 = 0$$

$$9K - 1 - 8K - 3 + 2K + 2 = 0$$

$$3K - 2 = 0$$

$$K = \frac{2}{3}$$

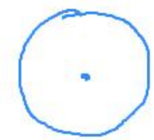
$$\frac{2}{3} : 1$$

$$\boxed{2:3} \quad +ve$$

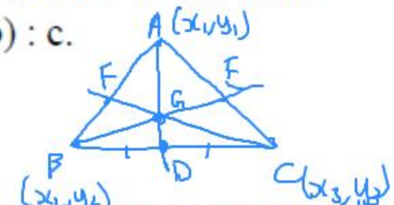
CENTROID AND INCENTRE :

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the coordinates of the centroid are : **
$$\boxed{\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)}$$

& the coordinates of the incentre are : ✓
$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$



Note that incentre divides the angle bisectors in the ratio $(b+c) : a$; $(c+a) : b$ & $(a+b) : c$.



Centroid → median.
Incentre → Angle Bisectors.
Circumcentre → ⊥ bisectors
Orthocentre → altitudes
Excentre → one internal bisector/ 2 external bis.

- For isosceles triangle centroid, circumcentre, orthocentre and incentre are collinear.
- For a triangle Orthocentre (O), Centroid (G), Circumcentre (C) are collinear and centroid divides orthocentre and circumcentre in the ratio 2 : 1 internally.

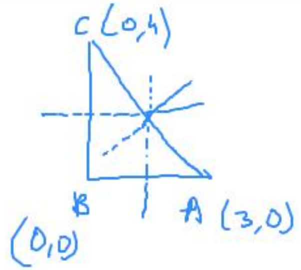
- For equilateral Δ, centroid, circumcentre, orthocentre and incentre coincide.

Problems

⊥ bisector

The circumcentre of the triangle with vertices (0, 0), (3, 0) and (0, 4) is -

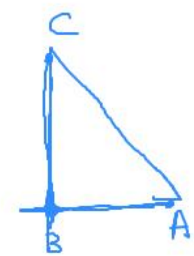
(A) (1, 1) (B) (2, 3/2) (C) (3/2, 2) (D) none of these



For a right angle Δ, circumcentre is the mid point of hypotenuse

$\frac{3+0}{2}, \frac{0+4}{2}.$

$\left(\frac{3}{2}, 2\right)$



Orthocentre = ?

B is the orthocentre.

equilateral Δ

All centres coincide

AREA OF A TRIANGLE :

The area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

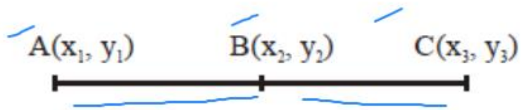
$$\frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right| = \frac{1}{2}\left\|\begin{matrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{matrix}\right\|$$

COLLINEARITY OF THREE POINTS :

Different conditions for three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ to be collinear are as follows

(i) $AB + BC = AC$, $AC - AB = BC$

(ii)



AB+BC=AC   dist formula.

Slope of AB = Slope of BC

$$\Rightarrow \quad \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

(iii) If the area of triangle ABC be zero then the three points will be collinear.

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Area of $\Delta = 0$

Problems

Find the value of x so that the points $\overset{A}{(x, -1)}$, $\overset{B}{(2, 1)}$ and $\overset{C}{(4, 5)}$ are collinear.

slope of AB = slope of BC

Area of $\Delta = 0$

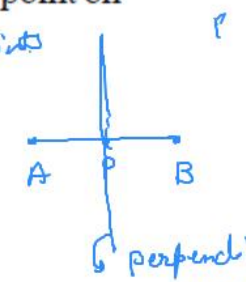
Locus of a point which is always at a fixed dist from a fixed point

LOCUS :

The curve described by a point which moves under given condition or conditions is called its locus.

EQUATION TO LOCUS OF A POINT :

The equation to the locus of a point is the relation which is satisfied by the coordinates of every point on the locus of the point. locus of a point such that it is always at equal distance from 2 fixed points



perpendicular bisector.

Steps to find locus of a point.

Step I : Assume the coordinates of the point say (h, k) whose locus is to be determined.

Step II: Write the given condition in mathematical form involving h, k.

Step III : Eliminate the variable (s), if any.

Step IV : Replace h by x and k by y in the result obtained in step III.
The equation so obtained is the locus of the point which moves under some condition(s).

EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS :

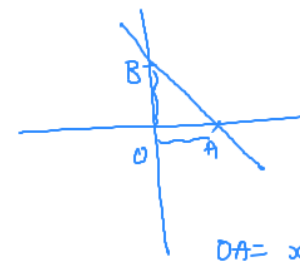
(i) **Slope – intercept form:** $\boxed{y = mx + c}$ → y- intercept

is the equation of a straight line whose slope is m & which makes an intercept c on the y–axis .

(ii) **Slope one point form:** $y - y_1 = m\ (x - x_1)$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1).

$A(x_1, y_1)$ m.

$y - y_1 = m(x - x_1)$



OA = x-int (+ve or -ve)

OB = y-int. (+ve or -ve)

(iv) **Two point form :** $\boxed{y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$ is the equation of a straight line which passes through the points (x_1, y_1) & (x_2, y_2).

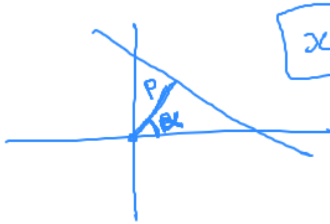
$A(x_1, y_1)$ $B(x_2, y_2)$ $\Rightarrow$ $y - y_1 = m(x - x_1)$

$y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

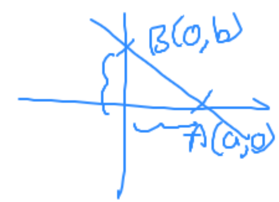
(v) **Intercept form :** $\dfrac{x}{a} + \dfrac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively.

(vi) **Perpendicular form :** $x\cos\alpha + y\sin\alpha = p$ is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes angle α with positive side of x–axis.

(vii) **General Form :** $\boxed{ax + by + c = 0}$ is the equation of a straight line in the general form



$\boxed{x\cos\alpha + y\sin\alpha = p}$



(iii) **Parametric form :** The equation of the line in parametric form is given by $\boxed{\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta}} = r$ (say). Where 'r' is the distance of any point (x , y) on the line from the fixed point (x_1, y_1) on the line. r is positive if the point (x, y) is on the right of (x_1, y_1) and negative if (x, y) lies on the left of (x_1, y_1) .

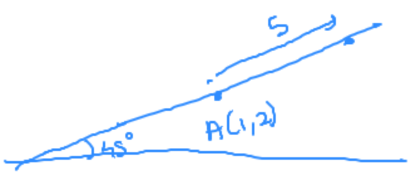
$y - y_1 = m(x - x_1)$

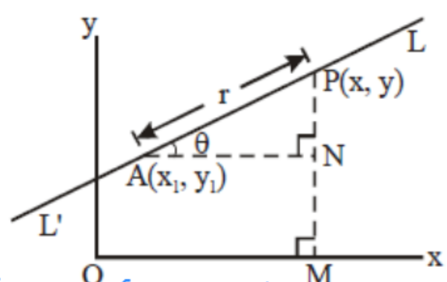
$y - y_1 = \tan\theta(x - x_1)$

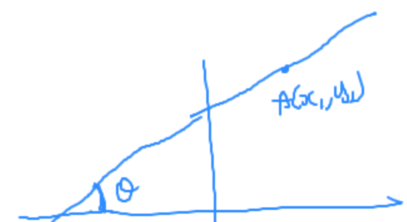
$y - y_1 = \frac{\sin\theta}{\cos\theta}(x - x_1) \Rightarrow \boxed{\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta}}$

$$\boxed{\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = \pm r}$$

where, r is the distance of any point on the line from the given point $A(x_1, y_1)$.







$m = \tan\theta$

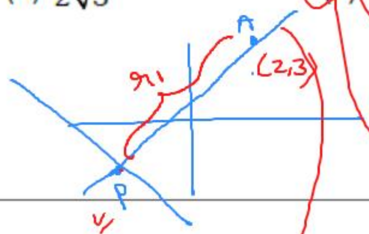
$\frac{x-1}{\cos 45^\circ} = \frac{y-2}{\sin 45^\circ} = \pm r \Rightarrow \left(x = 1 + \frac{r}{\sqrt{2}}, y = 2 + \frac{r}{\sqrt{2}}\right)$ r is parameter

$\left(1+\frac{5}{\sqrt{2}}, 2+\frac{5}{\sqrt{2}}\right)$ $\left(1-\frac{5}{\sqrt{2}}, 2-\frac{5}{\sqrt{2}}\right)$

Problems

Equation of a line which passes through point A(2, 3) and makes an angle of 45° with x axis. If this line meet the line $x + y + 1 = 0$ at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$



$y - 3 = \tan 45^\circ (x - 2)$

$y - 3 = x - 2$

$x - y + 1 = 0$

M-2

$\frac{x-2}{\cos 45^\circ} = \frac{y-3}{\sin 45^\circ} = r \Rightarrow \left(x = 2 + \frac{r_1}{\sqrt{2}},\ y = 3 + \frac{r_1}{\sqrt{2}}\right) \rightarrow P$

$x + y + 1 = 0$ (P satisfies)

$2 + \frac{r_1}{\sqrt{2}} + 3 + \frac{r_1}{\sqrt{2}} + 1 = 0 \Rightarrow \frac{2r_1}{\sqrt{2}} = -6 \Rightarrow r_1 = \frac{-6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = |-3\sqrt{2}|$

$= 3\sqrt{2}$

Problems

Equation of a line which passes through point A(2, 3) and makes an angle of 45° with x axis. If this line meet the line x + y + 1 = 0 at point P then distance AP is -

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $5\sqrt{2}$ (D) $2\sqrt{5}$

$y - 3 = \tan 45°(x - 2)$

$y - 3 = x - 2$

$x - y + 1 = 0$

M-2

$$\boxed{\frac{x-2}{\cos 45°} = \frac{y-3}{\sin 45°} = r} \Rightarrow \left(x = 2 + \frac{r_1}{\sqrt{2}},\ y = 3 + \frac{r_1}{\sqrt{2}}\right) \rightarrow P$$

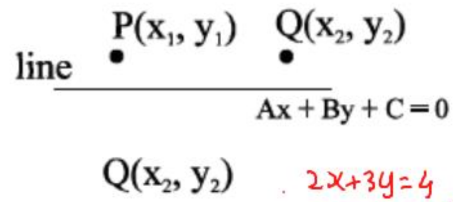
$x + y + 1 = 0$ (P satisfies)

$$2 + \frac{r_1}{\sqrt{2}} + 3 + \frac{r_1}{\sqrt{2}} + 1 = 0 \Rightarrow \frac{2r_1}{\sqrt{2}} = -6 \Rightarrow r_1 = \frac{-6}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = |-3\sqrt{2}|$$

$$= 3\sqrt{2}$$

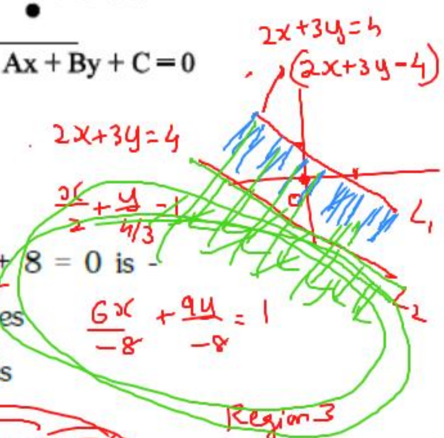
POSITION OF A POINT W.R.T. A LINE :

If the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ lies on the same side of the line $Ax + Bx + C$ then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of the same sign and if (x_1, y_1) and (x_2, y_2) are on the opposite side of the line then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of opposite sign.



$2x+3y=4$

parallel



The position of the point (8,–9) with respect to the lines $2x + 3y - 4 = 0$ and $6x + 9y + 8 = 0$ is -

(A) point lies on the same side of the lines
(B) point lies on one of the lines
(C) point lies on the different sides of the line
(D) point lies between the lines

$\frac{x}{2} + \frac{y}{4/3} = 1$

$\frac{6x}{-8} + \frac{9y}{-8} = 1$

	(0,0)	(8,-9)
$2x+3y-4$	$-$	$-$
$6x+9y+8$	$+$	$-$

$48-81+8$

Region-1 +

Region No 3

Region-2 −

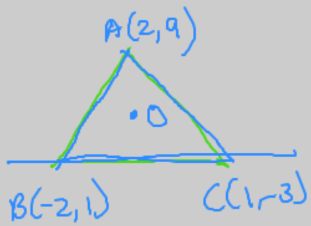
Ax+By+C

Same sign

Region 1

Problems

If (2, 9), (– 2, 1) and (1, –3) are the vertices of a triangle, then prove that the origin lies inside the triangle.



Exact Figure (M-1)

A & O should lie on same side of BC
B & O " " " " AC
C & O " " " " AB

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LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

The length of perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$.

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ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES :

If m_1 & m_2 are the slopes of two intersecting straight lines $(m_1 m_2 \neq -1)$ & θ is the acute angle between them, then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$.

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

Lines are parallel $m_1 = m_2$

Lines are $\perp$ $m_1 m_2 = -1$

Problems

If the straight line $3x + 4y + 5 - k(x + y + 3) = 0$ is parallel to y-axis, then the value of k is -

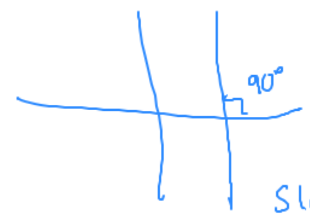
(A) 1 (B) 2 (C) 3 (D) 4

$(3-k)x + (4-k)y + 5 - 3k = 0$

$m = -\dfrac{(3-k)}{4-k} = \dfrac{k-3}{4-k}$

$4 - k = 0$

$k = 4.$



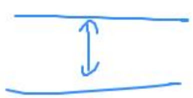
Slope $= \tan 90° = \infty$

PARALLEL LINES :

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$. Where k is a parameter.

(ii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ & $ax + by + c_2 = 0$ is $\left|\frac{c_1 - c_2}{\sqrt{a^2 + b^2}}\right|$.

Note that the coefficients of x & y in both the equations must be same.



$ax+by+c=0$

$ax+by+k=0$

$3x-4y+6=0$

$3x-4y+k=0$

PERPENDICULAR LINES :

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.

(ii) Straight lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ are at right angles if & only if $aa' + bb' = 0$.

$ax+by+c=0$

$m_1 m_2 = -1$

$2x+3y+5=0$

$\boxed{bx-ay+k=0}$ is $\perp$ to given line

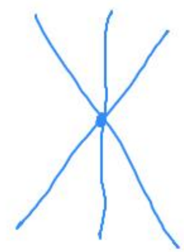
$\perp$ line $3x-2y+k=0$

CONCURRENCY OF THREE LINES :

Three lines are said to be concurrent if they pass through a common point, i.e. they meet at a point. Thus, if three line are concurrent the point of intersection of two lines lies on the third line. Let the three concurrent lines be

$$a_1x + b_1y + c_1 = 0 \qquad \ldots(i)$$
$$a_2x + b_2y + c_2 = 0 \qquad \ldots(ii)$$
$$a_3x + b_3y + c_3 = 0 \qquad \ldots(iii)$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$



Concurrent

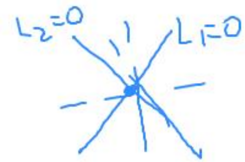
This is the required condition of concurrency of three lines.

Problems

Find the value of λ, if the lines $3x - 4y - 13 = 0$, $8x - 11y - 33 = 0$ and $2x - 3y + \lambda = 0$ are concurrent.

$$\begin{vmatrix} 3 & -4 & -13 \\ 8 & -11 & -33 \\ 2 & -3 & \lambda \end{vmatrix} = 0$$

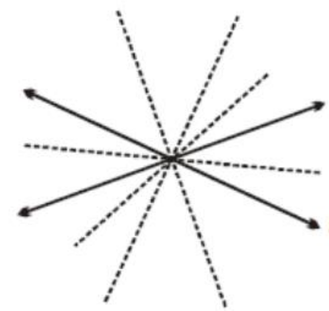
Solve & find λ



FAMILY OF STRAIGHT LINES :

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$

Then, the general equation of any straight line passing through the point of intersection of lines L_1 and L_2 is given by $L_1 + \lambda L_2 = 0$, where $\lambda \in R$



These lines form a family of straight line

Also this general equation satisfies point of intersection of L_1 and L_2 for any value of λ.

Conversely, if a variable line is expressed in the form of $\boxed{L_1 + \lambda L_2 = 0}$ $(\lambda \in R)$ then it always passes through fixed point which is the point of intersection of $L_1 = 0$ and $L_2 = 0$.

$3x+2y-5=0$, $x+y+1=0$

λ is parameter

$3x+2y-5+\lambda(x+y+1)=0$ $\quad \boxed{(3+\lambda)x+(2+\lambda)y-5+\lambda=0}$

slope = 5

Problems

Find the equation of the line through the point of intersection of the lines 3x − 4y + 1 = 0 & 5x + y − 1 = 0 and perpendicular to the line 2x − 3y = 10. — (2)

$m = \frac{-2}{-3} = \frac{2}{3}$

$m = -\frac{\text{coeff of } x}{\text{coeff of } y}$

$\rightarrow 3x - 4y + 1 + \lambda(5x + y - 1) = 0$

$(3+5\lambda)x + (\lambda - 4)y + 1 - \lambda = 0$ — (1)

$m = -\frac{(3+5\lambda)}{\lambda - 4}$

(1) & (2) are $\perp$

$m_1 \times m_2 = -1$

$-\frac{(3+5\lambda)}{\lambda - 4} \times \frac{2}{3} = -1$

$6 + 10\lambda = 3\lambda - 12 \Rightarrow 7\lambda = -18$

$\lambda = -18/7$

Ans. $\boxed{3x - 4y + 1 - \frac{18}{7}(5x + y - 1) = 0}$

Problems

Find the equation of the line through the point of intersection of the lines 3x – 4y + 1 = 0 & 5x + y – 1 = 0 and perpendicular to the line 2x – 3y = 10. — (2)

$m = \frac{-2}{-3} = \frac{2}{3}$

$m = -\left(\frac{\text{coeff of } x}{\text{coeff of } y}\right)$

$\rightarrow 3x - 4y + 1 + \lambda(5x + y - 1) = 0$

$(3+5\lambda)x + (\lambda - 4)y + 1 - \lambda = 0$ — (1)

$m = -\frac{(3+5\lambda)}{\lambda - 4}$

(1) & (2) are ⊥

$m_1 \times m_2 = -1$

$-\frac{(3+5\lambda)}{\lambda-4} \times \frac{2}{3} = -1$

$6 + 10\lambda = 3\lambda - 12 \Rightarrow 7\lambda = -18$

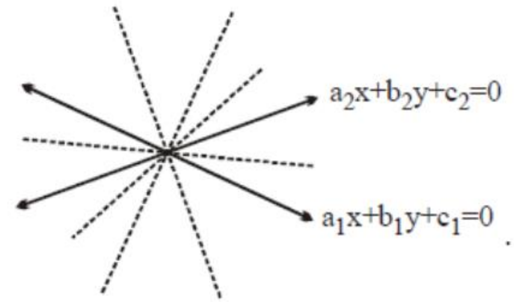
$\lambda = -18/7$

Ans. $\boxed{3x - 4y + 1 - \frac{18}{7}(5x + y - 1) = 0}$

FAMILY OF STRAIGHT LINES :

Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$

Then, the general equation of any straight line passing through the point of intersection of lines L_1 and L_2 is given by $L_1 + \lambda L_2 = 0$, where $\lambda \in R$



These lines form a family of straight line

Also this general equation satisfies point of intersection of L_1 and L_2 for any value of λ.

x-y

Conversely, if a variable line is expressed in the form of $\boxed{L_1 + \lambda L_2 = 0}$ $(\lambda \in R)$ then it always passes through fixed point which is the point of intersection of $L_1 = 0$ and $L_2 = 0$.

$\underbrace{x+y-1=0 \quad \& \quad 2x+3y+5=0}$ $\quad$ $\boxed{x+y-1+\lambda(2x+3y+5)=0}$ ✓

& also passing through (1,1)

Problems

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The line $(p + 2q)x + (p - 3q)y = p - q$ for different values of p and q passes through a fixed point whose co-ordinates are -

(A) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (B) $\left(\frac{2}{5}, \frac{2}{5}\right)$ (C) $\left(\frac{3}{5}, \frac{3}{5}\right)$ (D) $\left(\frac{2}{5}, \frac{3}{5}\right)$

$x+2y-1=0$ & $x-y+3=0$

$\boxed{L_1 + \lambda L_2 = 0}$

it will pass through p.o.i of L_1 & L_2

$x+y-1=0$
$2x-3y+1=0$ } p.o.i

$2x+2y-2=0$

$-5y+3=0 \Rightarrow y = 3/5, x = \frac{2}{5}$

$px + 2qx + py - 3qy = p - q$

$p(x+y-1) + q(2x-3y+1) = 0$

Divide by p

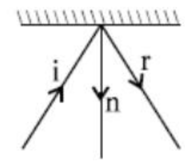
$(x+y-1) + \frac{q}{p}(2x-3y+1) = 0$

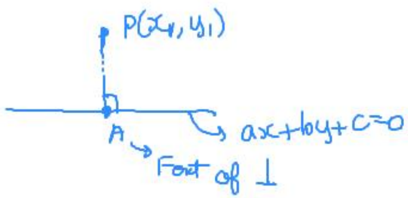
$L_1 + \lambda L_2 = 0$

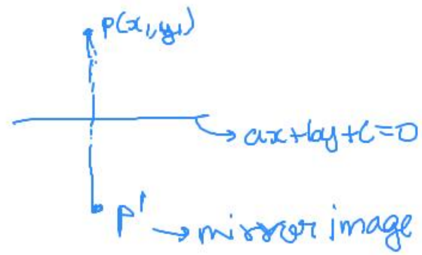
Optics Based Problem :

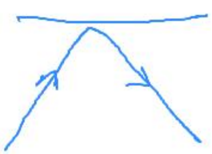
(i) Image of a point in a line.

(ii) A ray of light incident along the line $lx + my + n = 0$ and strikes a line mirror. if the equation of normal on the line mirror at the point of incidence is $px + qy + r = 0$ then find the equation of the reflected ray

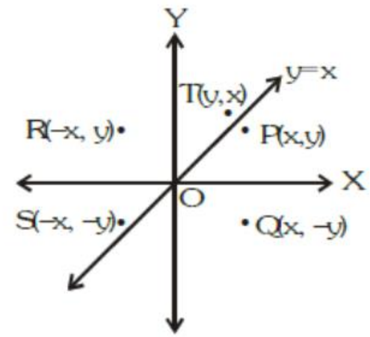








REFLECTION OF A POINT :



Let P(x, y) be any point, then its image with respect to

(a) x-axis is Q(x, −y)

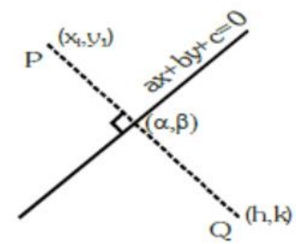
(b) y-axis is R(−x, y)

(c) origin is S(−x,−y)

(d) line y = x is T(y, x)

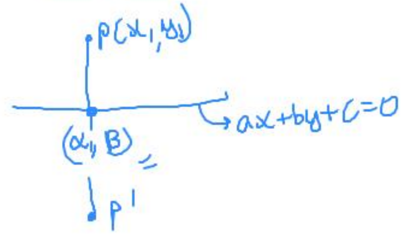
(e) Reflection of a point about any arbitrary line : The image (h,k) of a point $P(x_1, y_1)$ about the line $ax + by + c = 0$ is given by following formula.

$$\frac{h - x_1}{a} = \frac{k - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$



and the foot of perpendicular (α,β) from a point (x_1, y_1) on the line $ax + by + c = 0$ is given by following formula.

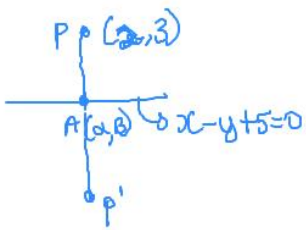
***
$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$



Problems

A ray of light is sent along the line $x - 2y - 3 = 0$. Upon reaching the line mirror $3x - 2y - 5 = 0$, the ray is reflected from it. Find the equation of the line containing the reflected ray.

$$\frac{\alpha - x_1}{a} = \frac{\beta - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}$$



$$\frac{\alpha - 2}{1} = \frac{\beta - 3}{-1} = \frac{-2(2 - 3 + 5)}{1^2 + (-1)^2}$$

$$\frac{\alpha - 2}{1} = \frac{\beta - 3}{-1} = -2 \times 2$$

$\alpha = -2$

$\beta = 7$

$(-2, 7) \rightarrow$ mirror image.

$\alpha = 0$, $\beta = 5$

$(0, 5)$ Foot of $\perp$

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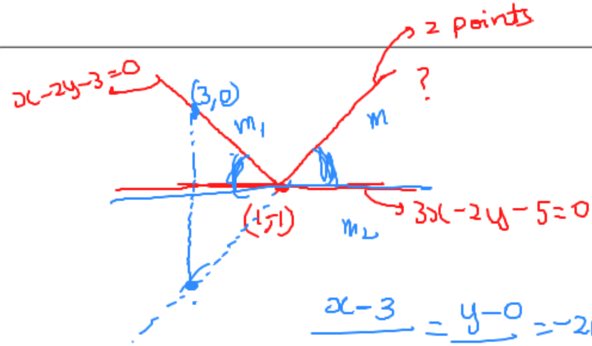
Problems

A ray of light is sent along the line x – 2y – 3 = 0. Upon reaching the line mirror 3x – 2y – 5 = 0, the ray is reflected from it. Find the equation of the line containing the reflected ray.



$$x-2y-3=0$$
$$3x-2y-5=0$$
$$-2x+2=0$$
$$x=1,\ y=-1$$

We got 2 points on reflected ray $(1,-1)$ & $\frac{15}{13}, \frac{16}{13}$

$$\left|\frac{m_1-m_2}{1+m_1m_2}\right| = \left|\frac{m-m_2}{1+m m_2}\right|$$

$$\frac{x-3}{3} = \frac{y-0}{-2} = \frac{-2(9-5)}{3^2+2^2}$$

$$\frac{x-3}{3} = \frac{y}{-2} = \frac{-8}{13} \Rightarrow x = 3-\frac{24}{13},\ y=\frac{16}{13}$$

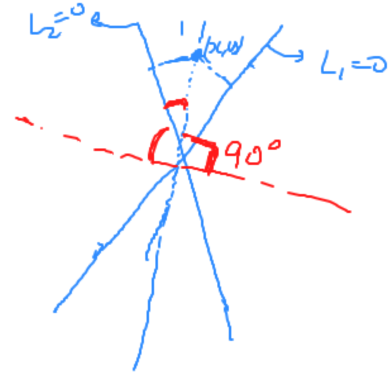
~~x = 1/11, y = 16/11~~ $\left(\frac{15}{13}, \frac{16}{13}\right)$

BISECTORS OF THE ANGLES BETWEEN TWO LINES : $L_1=0$

Equations of the bisectors of angles between the lines $ax+by+c=0$ & $a'x+b'y+c'=0$ $(ab' \neq a'b)$ are : $\frac{ax+by+c}{\sqrt{a^2+b^2}} = \pm \frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

$L_2=0$

$$\left|\frac{ax+by+c}{\sqrt{a^2+b^2}}\right| = \left|\frac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}\right|$$



Angle bisector is equidistance from both lines

$x-y+2=0$ & $3x+2y+6=0$

$$\frac{x-y+2}{\sqrt{2}} = \pm \frac{3x+2y+6}{\sqrt{13}}$$

Any two lines have 2 bisectors which are $\perp$ to each other

Find acute angle bisector

obtuse angle bisector

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To discriminate between the acute angle bisector & the obtuse angle bisector

To discriminate between acute angle bisector & obtuse angle bisector proceed as follows Write $ax+by+c=0$ & $a'x+b'y+c'=0$ such that constant terms are positive.

If $\boxed{aa'+bb'}<0$, then the angle between the lines that contains the origin is acute and the equation of the

bisector of this acute angle is $\dfrac{ax+by+c}{\sqrt{a^2+b^2}}=+\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

$x-y-2=0 \Rightarrow -x+y+2=0$

$2x+3y+5=0$

$\boxed{\dfrac{-x+y+2}{\sqrt{2}}=\pm\dfrac{2x+3y+5}{\sqrt{13}}}$

$aa'+bb'>0$

$+\rightarrow$ obtuse $-$ (acute)

$aa'+bb' = (-1)(2)+(1)(3)=+1>0$

obtuse bisector

If, however, $aa'+bb'>0$, then the angle between the lines that contains the origin is obtuse & the equation of the bisector of this obtuse angle is:

$\dfrac{ax+by+c}{\sqrt{a^2+b^2}}=+\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$; therefore $\dfrac{ax+by+c}{\sqrt{a^2+b^2}}=-\dfrac{a'x+b'y+c'}{\sqrt{a'^2+b'^2}}$

is the equation of other bisector.

If $aa'+bb'>0$ then $+$ gives obtuse & $-$ gives acute

$aa'+bb'<0$ then $-$ gives obtuse & $+$ gives acute

Pair of Straight Lines

A PAIR OF STRAIGHT LINES THROUGH ORIGIN :

A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin & if :

(a) $h^2 > ab \Rightarrow$ lines are real & distinct .

(b) $h^2 = ab \Rightarrow$ lines are coincident .

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)

$x+y+5=0$

$2x+3y-1=0$

$\boxed{(x+y+5)(2x+3y-1)=0}$

$\boxed{2^{nd} \text{ degree eqn.}}$

If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$m_1 + m_2 = -\dfrac{2h}{b}$ & $m_1 m_2 = \dfrac{a}{b}$.

2^{nd} degree eqn in x & y

$ax^2+2hxy+by^2+2gx+2fy+c=0$

→ pair of st lines
→ circle
→ parabola
→ ellipse
→ hyperbola

depends on coefficients

a, h, b, g, f, c

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General Equation Of Second Degree Representing A Pair Of Straight Lines:

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

pair of straight lines

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only .

$ax^2+2hxy+by^2+2gx+2fy+c$

Problems

If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then λ is equal to -

(A) 4 (B) 3 (C) 2 (D) 1

$a=\lambda$, $h=-5$, $b=12$, $g=\frac{5}{2}$, $f=(-8)$, $c=-3$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda & -5 & 5/2 \\ -5 & 12 & -8 \\ \frac{5}{2} & -8 & -3 \end{vmatrix} = 0$$

$\lambda = 2$

A PAIR OF STRAIGHT LINES THROUGH ORIGIN :

A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin & if :

(a) $h^2 > ab \Rightarrow$ lines are real & distinct .

(b) $h^2 = ab \Rightarrow$ lines are coincident .

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)

If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

→ Homogeneous eqn of 2^{nd} degree

$|g \ f \ c| = |0 \ 0 \ 0| = 0$

$\boxed{ax^2 + 2hxy + by^2 = 0}$ ** $g = 0, f = 0, c = 0$

→ will always represent pair of st line

A PAIR OF STRAIGHT LINES THROUGH ORIGIN :

Very easy

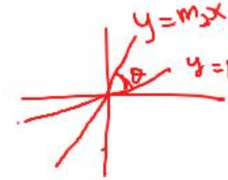
A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin & if :

$(m_1x - y)(m_2x - y) = 0$

(a) $h^2 > ab \Rightarrow$ lines are real & distinct .

(b) $h^2 = ab \Rightarrow$ lines are coincident .

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)



$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \frac{\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1+m_1m_2} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1+\frac{a}{b}} = \boxed{\frac{2\sqrt{h^2-ab}}{a+b}}$

$ax^2 + 2hxy + by^2$

$y = m_1x \quad y = m_2x$

If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then:

$\frac{y}{x} = m_1 \quad \frac{y}{x} = m_2$

$$m_1 + m_2 = -\frac{2h}{b} \ \& \ m_1 m_2 = \frac{a}{b}.$$

Divide by x^2

$a + 2h\frac{xy}{x^2} + b\frac{y^2}{x^2} = 0$

** $\boxed{bm^2 + 2hm + a = 0}$ ** $\to m_1$, $\to m_2$

$\boxed{\begin{array}{l} m_1 + m_2 = -2h/b \\ m_1m_2 = a/b \end{array}}$

$\frac{y}{x} = m$

$\boxed{b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0}$

If θ is the acute angle between the pair of straight lines represented by,

$ax^2 + 2hxy + by^2 = 0$, then; $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right|$.

If $h^2 = ab \Rightarrow \tan\theta = 0 \Rightarrow$ lines are coincident

The condition that these lines are:

(a) At right angles to each other is $a + b = 0$. i.e. co–efficient of x^2 + coefficient of $y^2 = 0$.

(b) Coincident is $h^2 = ab$.

(c) Equally inclined to the axis of x is $\boxed{h = 0.}$ i.e. coeff. of $xy = 0$.

$x^2 + 7xy - y^2 = 0$

$a = 1$

$b = -1$

$a + b = 0$

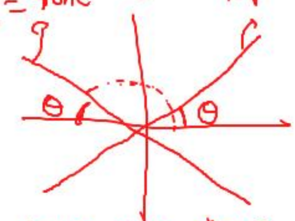
$\tan\theta = \tan 90° = $ N.D

$\dfrac{2\sqrt{h^2 - ab}}{a+b}$

$\boxed{a + b = 0}$ **

↳ lines to be ⊥

$m_2 = \tan(\pi - \theta)$ $m_1 = \tan\theta$



$m_1 + m_2 = -\dfrac{2h}{b} = 0 \Rightarrow \boxed{h = 0}$

$m_1 = \tan\theta$, $m_2 = -\tan\theta$

$m_1 + m_2 = 0$

Problems

Prove that the equation $x^2 - 5xy + 4y^2 = 0$ represents two lines passing through the origin. Also find their equations.

$s=0$

$x^2 - 5xy + 4y^2 = 0$ Divide by x^2

$1 - 5\frac{y}{x} + 4\left(\frac{y}{x}\right)^2 = 0$ $\frac{y}{x} = m.$ $y = mx$

$1 - 5m + 4m^2 = 0$

$4m^2 - 5m + 1 = 0$ $m_1 = \frac{1}{4}, m_2 = 1$

$4m^2 - 4m - m + 1 = 0$ $y = m_1 x$ $y = m_2 x$

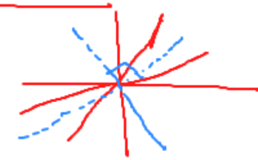
$(4m-1)(m-1) = 0$ $y = \frac{1}{4}x, y = 1x$ $\Rightarrow y = \frac{x}{4}, y = x$

$m = \frac{1}{4}, 1$

The equation to the straight lines bisecting the angle between the straight lines,

$ax^2 + 2hxy + by^2 = 0$ is $\boxed{\dfrac{x^2 - y^2}{a - b} = \dfrac{xy}{h}}$.

$y = m_1 x$ $\quad y = m_2 x$



$$\boxed{\frac{x^2 - y^2}{a - b} = \frac{xy}{h}.}$$

The product of the perpendiculars, dropped from (x_1, y_1) to the pair of lines represented by the equation, $ax^2 + 2hxy + by^2 = 0$ is $\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$.

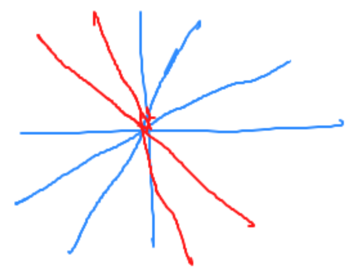
Pair of straight lines perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$ and through origin are given by $bx^2 - 2hxy + ay^2 = 0$.

$3x^2 - 2xy + 5y^2 = 0$

$3x + 4y + 5 = 0$

$\boxed{4x - 3y + \lambda = 0}$

$\boxed{5x^2 + 2xy + 3y^2 = 0}$



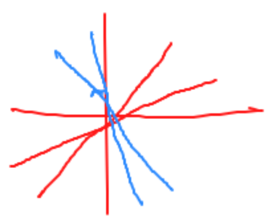
Problems

Equation of the pair of straight lines through origin and perpendicular to the pair of straight lines $5x^2 - 7xy - 3y^2 = 0$ is -

(A) $3x^2 - 7xy - 5y^2 = 0$ (B) $3x^2 + 7xy + 5y^2 = 0$ (C) $3x^2 - 7xy + 5y^2 = 0$ (D) $3x^2 + 7xy - 5y^2 = 0$

$-3x^2 + 7xy + 5y^2 = 0 \Rightarrow 3x^2 - 7xy - 5y^2 = 0$

$ax^2 + 2hxy + by^2 = 0$



$\perp$ lines: $bx^2 - 2hxy + ay^2 = 0$

If θ is the acute angle between the pair of straight lines represented by,

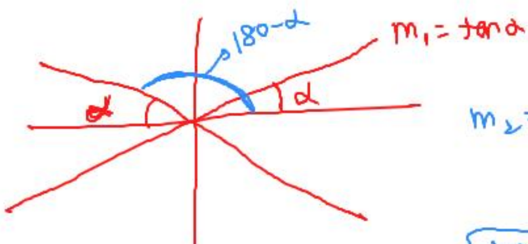
$ax^2 + 2hxy + by^2 = 0$, then; $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right|$.

The condition that these lines are:

(a) At right angles to each other is $a + b = 0$. i.e. co–efficient of x^2 + coefficient of $y^2 = 0$.

(b) Coincident is $h^2 = ab$.

(c) ~~Equally inclined to the ax~~is of x is $h = 0$. i.e. coeff. of $xy = 0$.



$m_2 = \tan(180 - \alpha)$

$= -\tan\alpha$

$\boxed{m_1 + m_2 = 0}$

$-\dfrac{2h}{b} = 0 \Rightarrow h = 0$

$ax^2 + 2hxy + by^2 = 0$ $\to y = m_1 x$, $\to y = m_2 x$

$y = mx$

$\boxed{\dfrac{y}{x} = m}$

Divide by x^2

$a + 2h\dfrac{y}{x} + b\left(\dfrac{y}{x}\right)^2 = 0$

$bm^2 + 2hm + a = 0$ $\to m_1$, $\to m_2$

$m_1 + m_2 = -\dfrac{2h}{b}$, $m_1 m_2 = \dfrac{a}{b}$