PHYSICS

NEET and JEE Main 2020 : 45 Days Crash Course

Rotational Motion Part 02

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Angular Momentum

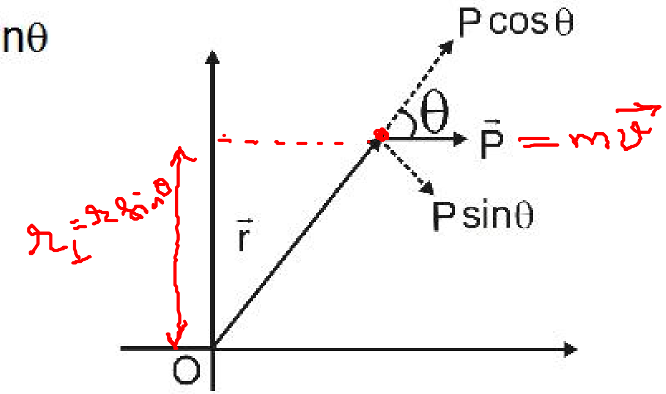
vector

$\vec{L} = \vec{r} \times \vec{P}$ $\Rightarrow$ $L = rp\sin\theta$

or $|\vec{L}| = r_{\perp} \times P$

or $|\vec{L}| = P_{\perp} \times r$

$\vec{\tau} = \vec{r} \times \vec{F}$



where $\vec{P}$ = momentum of particle

$\vec{r}$ = position of vector of particle with respect to point O about which angular momentum is to be calculated .

θ = angle between vectors $\vec{r}$ & $\vec{P}$

$r_{\perp}$ = perpendicular distance of line of motion of particle from point O.

D.F.

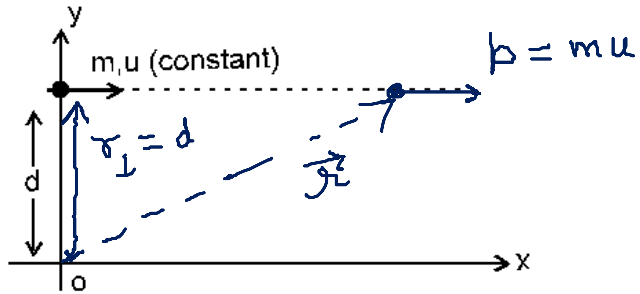
$[ML^2T^{-1}]$

$P_{\perp}$ = component of momentum perpendicular to $\vec{r}$.

SI unit of angular momentum is kgm^2/sec.

Example

A particle of mass 'm' starts moving from point (o,d) with a constant velocity u$\hat{i}$. Find out its angular momentum about origin at this moment what will be the answer at the later time?



Sol.

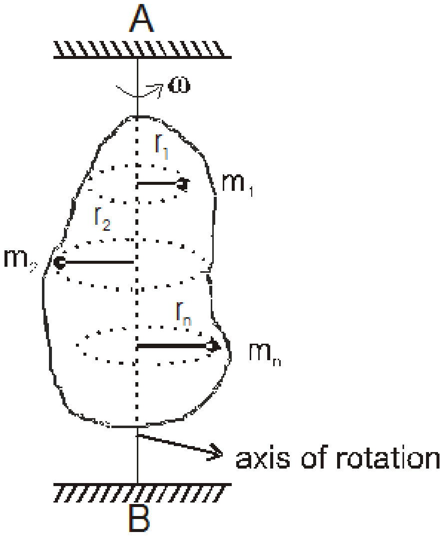
Initial

$$L = r_\perp p = d \cdot mu = mud(-\hat{k})$$

Later

$$L = r_\perp p = d \cdot mu = mud(-\hat{k})$$

Angular momentum of a rigid body rotating about fixed axis



$\boxed{L = I\,\omega}$

L = Angular momentum of object about axis of rotation.

I = Moment of Inertia of rigid , body about axis of rotation.

ω = angular velocity of the object.

For fixed axis rotation;

$$K.E. = \frac{1}{2} I\omega^2 = \frac{I^2\omega^2}{2I}$$

$$= \frac{L^2}{2I}$$

$$K.E = \frac{p^2}{2m}$$

Example

A uniform circular ring of mass 400 g and radius 10 cm is rotated about one of its diameter at an angular speed of 20 rad/s. Find the kinetic energy of the ring and its angular momentum about the axis of rotation.

Sol.

The moment of inertia of the circular ring about its diameter is

$$I = \frac{1}{2}Mr^2 = \frac{1}{2}\,(0.400\text{ kg})\,(0.10\text{ m})^2 = 2 \times 10^{-3}\text{ kg-m}^2.$$

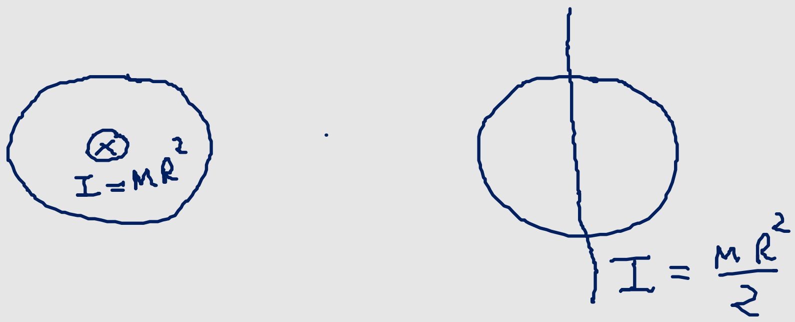
The kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}\,(2 \times 10^{-3}\text{ kg - m}^2)\,(400\text{ rad}^2/\text{s}^2) = 0.4\text{ J}$$

and the angular momentum about the axis of rotation is

$$L = I\omega = (2 \times 10^{-3}\text{ kg-m}^2)\,(20\text{ rad/s})$$

$$= 0.04\text{ kg-m}^2/\text{s} = 0.04\text{ J-s}.$$



Conservation of Angular Momentum

Newton's 2nd law in rotation : $\vec{\tau} = \frac{d\vec{L}}{dt}$

where $\vec{\tau}$ and $\vec{L}$ are about the same axis.

If $\tau_{ext} = 0$ about the axis of rotation.

then $\vec{L}$ = constant

Impulse of Torque : $\int \tau dt = \Delta l = \Delta J$

$\Delta l \rightarrow$ Charge in angular momentum.

$\vec{F} = \frac{d\vec{p}}{dt}$

$\int \vec{F} dt = \Delta \vec{p}$

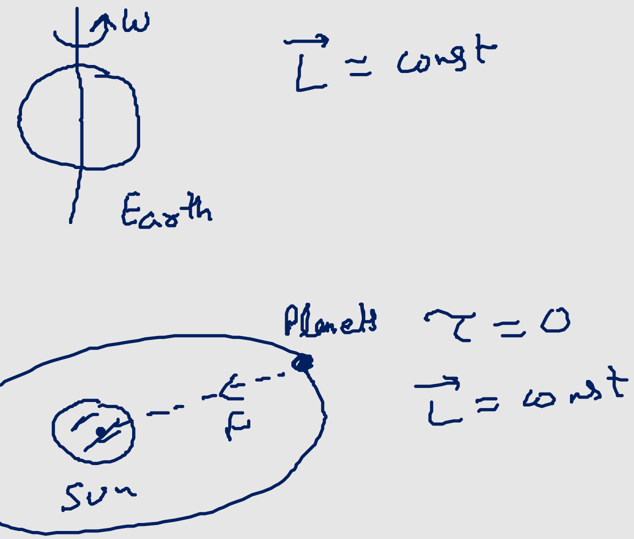
Example

Keeping the mass of earth constant. If its radius is halved then the duration of the day will be.

Sol.

$$I_1\omega_1 = I_2\omega_2 \Rightarrow \frac{2}{5}MR^2\frac{2\pi}{T_1} = \frac{2}{5}M\left(\frac{R}{2}\right)^2\frac{2\pi}{T_2}$$

$$\Rightarrow T_2 = \frac{T_1}{4} \because T_1 = 24\text{ hr} \qquad \therefore \boxed{T_2 = 6\text{ hr}}$$



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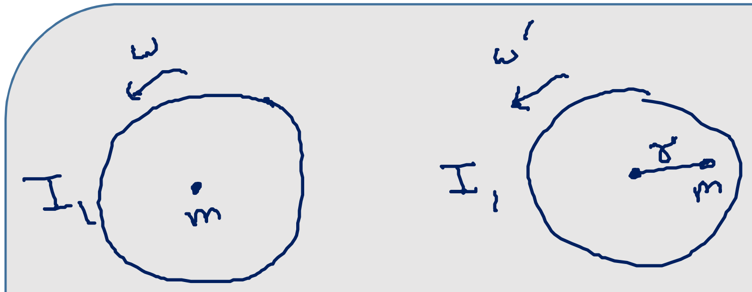
Example

A rotating table has angular velocity ' ω ' and moment of inertia I_1. A person of mass 'm' stands on centre of rotating table. If the person moves a distance r along its radius then what will be the final angular velocity of rotating table.

Sol.

Initial angular momentum = Final angular momentum

$$I_1\omega_1 = (I_1 + m\,r^2)\,\omega_2 \Rightarrow \omega_2 = \frac{I_1\omega_1}{I_1 + m\,r^2}$$



$$I_f\,\omega_f = I_i\,\omega_i$$

$$(I_1 + mr^2)\,\omega' = I_1\,\omega$$

$$\omega' = \frac{I_1\,\omega}{I_1 + mr^2}$$

Combined Translational and Rotational motion of a rigid body

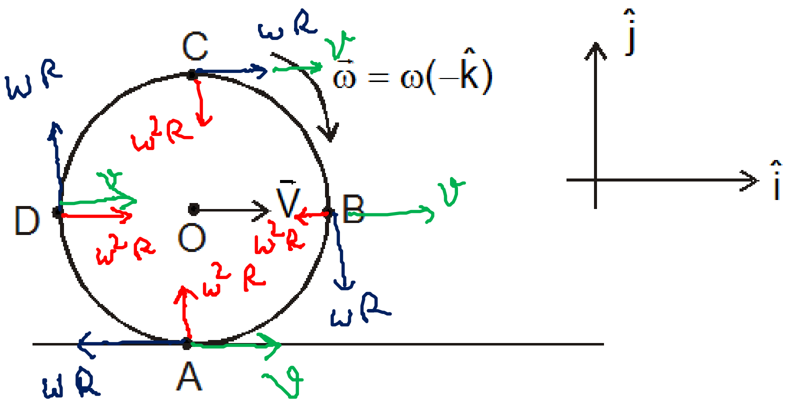
The general motion of a rigid body can be thought of as a sum of two independent motions. A translation of some point of the body plus a rotation about this point. A most convenient choice of the point is the centre of mass of the body as it greatly simplifies the calculations.

Consider the general motion of a wheel (radius r) which can be view on pure translation of its center O (with the velocity v) and pure rotation about O (with angular velocity ω)

$v = \omega R$

$a_r = \omega^2 R$

$a_t = \alpha R = 0$



vel. of A, B, C & D

acc. of A, B, C & D

$\vec{v}_A = (v - \omega R)\hat{i}$

$\vec{v}_B = v\hat{i} - \omega R\hat{j}$

$\vec{v}_C = (v + \omega R)\hat{i}$

$\vec{v}_D = v\hat{i} + \omega R\hat{j}$

$\vec{a}_A = \omega^2 R\hat{j}$

$\vec{a}_B = -\omega^2 R\hat{i}$

$\vec{a}_C = -\omega^2 R\hat{j}$

$\vec{a}_D = \omega^2 R\hat{i}$

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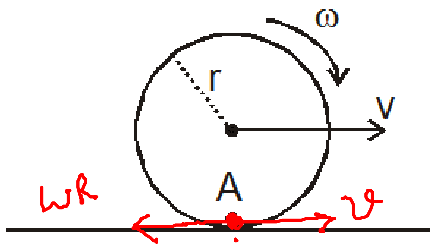
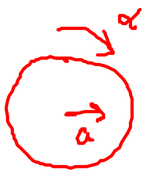
Pure Rolling (or rolling without sliding)

Pure rolling means there is no relative motion between the rolling body and the surface of contact, at the point of contact.





For pure rolling, velocity of A w.r.t. ground is zero.

$$\Rightarrow v - \omega r = 0$$

$$\boxed{v = \omega r}$$

for fixed surface

Similarly $\boxed{a = \alpha r}$

At pt. of contact

$\vec{v}_{rel} = 0$

$\vec{a}_{rel} = 0$ in tangential dirn

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Dynamics of general motion of a rigid body

This motion can be viewed as translation of centre of mass and rotation about an axis passing through centre of mass

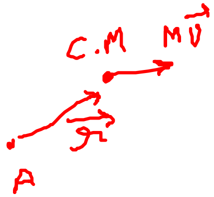
If

I_{CM} = Moment of inertia about this axis passing through COM

τ_{cm} = Net torque about this axis passing through COM

$\vec{a}_{CM}$ = Acceleration of COM

$\vec{v}_{CM}$ = Velocity of COM



$\vec{F}_{ext}$ = Net external force acting on the system.

$\vec{P}_{system}$ = Linear momentum of system.

$\vec{L}_{CM}$ = Angular momentum about centre of mass.

$\vec{r}_{CM}$ = Position vector of COM w.r.t. point A.

then

(i) $\boxed{\vec{\tau}_{cm} = I_{cm}\,\vec{\alpha}}$

(ii) $\vec{F}_{ext} = M\vec{a}_{cm}$

(iii) $\vec{P}_{system} = M\vec{v}_{cm}$

(vi) $\boxed{\text{Total K.E.} = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\,\omega^2}$

(v) $\vec{L}_{CM} = I_{CM}\,\vec{\omega}$

(vi) Angular momentum about point A

= $\vec{L}$ about C.M. + $\vec{L}$ of C.M. about A

$\vec{L}_A = I_{cm}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}$

Example

solid

A uniform sphere of mass 200 g rolls without slipping on a plane surface so that its centre moves at a speed of 2.00 cm/s. Find its kinetic energy.

Sol.

As the sphere rolls without slipping on the plane surface, its angular speed about the centre is $\omega = \frac{v_{cm}}{r}$. The kinetic energy is

$\omega = \frac{v}{r}$

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} Mv_{cm}^2 = \frac{1}{2} \cdot \frac{2}{5} Mr^2\omega^2 + \frac{1}{2} Mv_{cm}^2$$

$$= \frac{1}{5} Mv_{cm}^2 + \frac{1}{2} Mv_{cm}^2 = \frac{7}{10} Mv_{cm}^2 = \frac{7}{10} (0.200 \text{ kg}) (0.02 \text{ m/s})^2 = 5.6 \times 10^{-5} \text{ J}.$$

Example

A spherical ball rolls on a table without slipping. Then the fraction of its total energy associated with rotation is -

(A) 2/5 (B) 2/7 (C) 3/5 (D) 3/7

Sol. Total energy

$$K = K_R + K_T = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

$$= \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2 + \frac{1}{2} mr^2\omega^2$$

$$= \frac{1}{5} mr^2\omega^2 + \frac{1}{2} mr^2\omega^2 = \frac{7}{10} mr^2\omega^2$$

Now, rotational kinetic energy

$$K_R = \frac{1}{2} I\omega^2 = \frac{1}{5} mr^2\omega^2$$

$$\therefore \frac{K_R}{K} = \frac{\frac{1}{5}mr^2\omega^2}{\frac{7}{10}mr^2\omega^2} = \frac{2}{7}$$

$$\frac{R.K.E}{T.K.E}$$

$v = \omega r$

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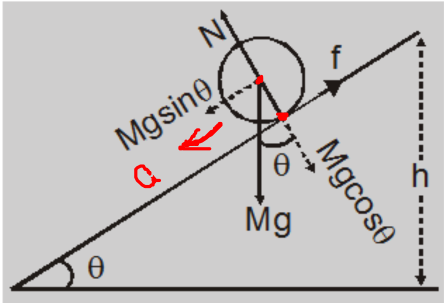
Rolling on Inclined Surface

A body of mass M and radius R rolling down a plane inclined at an angle θ with the horizontal. The body rolls without slipping.



$\vec{a} = \vec{\alpha} R$

Static friction

$$Ma = Mg\sin\theta - f \quad \text{— (1)}$$

torque acting on the body $\tau = I\alpha = fR$

$$\Rightarrow \quad f = \frac{I\alpha}{R} = \frac{Ia}{R^2} \qquad (a = \alpha R)$$

so we can write $\quad Ma = Mg\sin\theta - \dfrac{Ia}{R^2} \Rightarrow a = g\sin\theta - \dfrac{Ia}{MR^2}$

$$\text{or} \quad \boxed{a = \frac{g\sin\theta}{1 + \dfrac{I}{MR^2}}} \quad \Rightarrow \quad \boxed{a = \frac{g\sin\theta}{1 + \dfrac{K^2}{R^2}}}$$

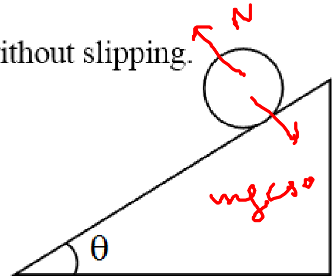
Radius of gyration (c.o.m.)

$$\Rightarrow \quad a = \frac{g\sin\theta}{1+n^2} \qquad \left[I = MK^2 \text{ and let } \frac{K^2}{R^2} = n^2\right]$$

frictional force f acting on the body : $f = Mg\sin\theta - M\dfrac{g\sin\theta}{1+\dfrac{K^2}{R^2}} = Mg\sin\theta\left[\dfrac{\dfrac{K^2}{R^2}}{1+\dfrac{K^2}{R^2}}\right] = \boxed{Mg\sin\theta\left[\dfrac{n^2}{1+n^2}\right]}$

Example

A spherical shell of radius R is rolling down an incline of inclination θ without slipping. Find minimum value of coefficient of friction -



(A) $\frac{2}{7}\tan\theta$ ✓(B) $\frac{2}{5}\tan\theta$

(C) $\frac{2}{3}\tan\theta$ (D) none

Sol.

$$f = mg\sin\theta\left[\frac{n^2}{1+n^2}\right] = mg\sin\theta\left[\frac{1}{\frac{1}{n^2}+1}\right]$$

$$n^2 = \frac{k^2}{R^2} = \frac{I_{cm}}{mR^2} = \frac{\frac{2}{3}mR^2}{mR^2} = \frac{2}{3}$$

$$f = mg\sin\theta\left(\frac{1}{\frac{3}{2}+1}\right) = \frac{2\,mg\sin\theta}{5}$$

$$f \le \mu N \Rightarrow \frac{2}{5}\cancel{m}g\sin\theta \le \mu\cdot\cancel{m}g\cos\theta$$

$$\mu \ge \frac{2}{5}\tan\theta$$

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Comparison between formula of translatory motion and rotatory motion

Translatory Motion	Rotatory Motion
$\vec{F} = \frac{d\vec{p}}{dt}$	$\vec{\tau} = \frac{d\vec{L}}{dt}$
$\vec{F} = m\vec{a}$	$\vec{\tau} = I\vec{\alpha}$
◉ Linear momentum ($\vec{P}$)	◉ Angular momentum ($\vec{L}$)
$\vec{P} = m\vec{v}$	$\vec{J} = I\vec{\omega}$
◉ Linear kinetic energy	◉ Rotational kinetic energy
$KE = \frac{1}{2}mv^2$	$E = \frac{1}{2}I\omega^2$
◉ Work done $W = \vec{F}.\vec{S}$ (constant force)	◉ Work done $W = \vec{\tau}.\vec{\theta}$ (constant torque)
◉ Variable force	◉ Variable torque
$W = \int \vec{F}.d\vec{s}$	$W = \int \vec{\tau}.d\vec{\theta}$
◉ Power in linear motion	◉ Power in rotational motion
$P = \frac{dW}{dt} = \frac{\vec{F}.d\vec{s}}{dt}$	$P = \frac{dW}{dt} = \frac{\vec{\tau}.d\vec{\theta}}{dt}$
$P = \vec{F}.\vec{V}$	$P = \vec{\tau}.\vec{\omega}$
◉ Work energy theorem in T. M.	◉ Work energy theorem in R. M.
$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$	$W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2$
◉ Linear impulse	◉ Angular impulse
It is product of large force for small time	It is product of large torque for small time
$\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$	$\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}$
$\Delta\vec{p} = \vec{F}\Delta t$ = Impulse	$\Delta\vec{L} = \vec{\tau}\Delta t$ = Angular impulse
Impulse momentum theorem	Angular Impulse momentum theorem