PHYSICS

NEET and JEE Main 2020 : 45 Days Crash Course

Rotational Motion Part 01

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Kinematics of Circular/Rotational Motion

Average Angular Velocity

$$\omega_{av} = \frac{\text{Angular displacement}}{\text{Total time taken}}$$

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

Average Angular Acceleration :

$$\alpha_{av} = \frac{\text{Change in angular velocity}}{\text{Total time taken}}$$

$$\vec{\alpha}_{av} = \frac{\vec{\omega}_2 - \vec{\omega}_1}{t_2 - t_1} = \frac{\Delta\vec{\omega}}{\Delta t}$$

Instantaneous Angular Acceleration :

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

since $\vec{\omega} = \frac{d\vec{\theta}}{dt}$, $\therefore \vec{\alpha} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$,

Also $\vec{\alpha} = \omega \frac{d\vec{\omega}}{d\theta}$

Instantaneous Angular Velocity

$$\vec{\omega} = \frac{d\vec{\theta}}{dt}$$

rad/s $[T^{-1}]$

rad/s^2 $[T^{-2}]$

$$v_{avg} = \frac{\Delta x}{\Delta t}$$

$$v = \frac{dx}{dt}$$

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t}$$

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$a = d^2x/dt^2$$

$$= v\frac{dv}{dx}$$

Motion with constant angular acceleration

$\omega_0 \rightarrow$ Initial angular velocity ; $\omega \rightarrow$ Final angular velocity

$\alpha \rightarrow$ Constant angular acceleration ; $\theta \rightarrow$ Angular displacement

Circular motion with constant angular acceleration is analogous to one dimensional translational motion with constant acceleration. Hence even here equation of motion have same form.

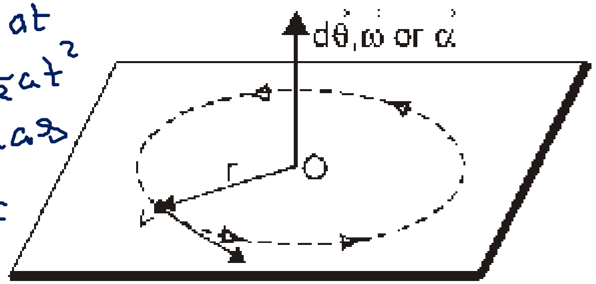
$$\omega = \omega_0 + \alpha t \qquad v = u + at$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad s = ut + \frac{1}{2}at^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \qquad v^2 = u^2 + 2as$$

$$\theta = \left(\frac{\omega + \omega_0}{2}\right)t \qquad s = \left(\frac{u+v}{2}\right)t$$

$$\theta_{n^{th}} = \omega_0 + \frac{\alpha}{2}(\theta_n - \theta_{n-1}) \qquad s_n = u + \frac{a}{2}(2n-1)$$



Example

A disc rotates with a uniform angular acceleration of 2.0 rad/s^2 about its axis. If the disc starts from rest, how many revolutions will it make in the first 10 seconds?

Sol.

The angular displacement in the first 10 seconds is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}(2.0\ \text{rad/s}^2)(10\ \text{s})^2 = 100\ \text{rad}.$$

As the wheel turns by 2π radian in each revolution,
the number of revolutions in 10 s is

$$n = \frac{100}{2\pi} = 16$$

$\alpha = 2\ \text{rad/s}^2$

$\omega_0 = 0$

$t = 10\ \text{s}$

$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = 0 + \frac{1}{2} \times 2 \times 10^2$

$= 100\ \text{rad}$

$1\ \text{rev.} = 2\pi\ \text{rad} = 360^\circ$

$\theta = \frac{100}{2\pi}\ \text{rev.} = \frac{50}{\pi}$

$= \frac{50}{22} \times 7 = \frac{175}{11}$

$= 16\ \text{rev.}$

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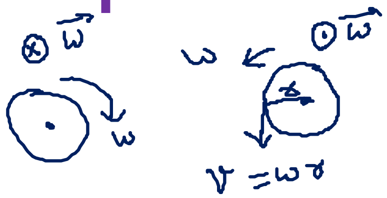
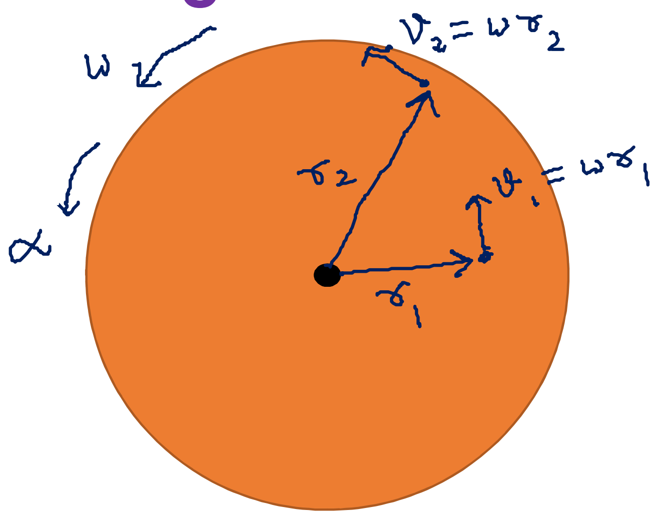
Relation between speed and angular velocity

For circular/rotational motion, $v = \omega r$

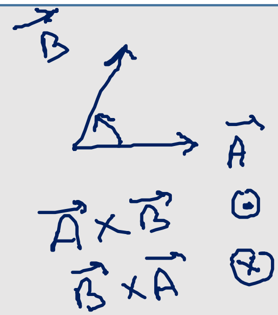
In general, $\vec{v} = \vec{\omega} \times \vec{r}$ ← Axial vector



For rotating body, angular velocity (ω) and angular acceleration (α) is same for all points but linear speed and linear acceleration will be different (more if r is more)



$\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$



linear vel. & acc.

$r \uparrow$ v & $a \uparrow$

Moment of Inertia

The moment of inertia of a particle with respect to an axis of rotation is equal to the product of mass of the particle and square of distance from rotational axis.

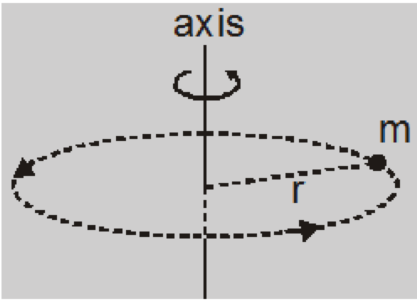
$$I = mr^2$$

Scalar

$I \geq 0$

$Kg\text{-}m^2$

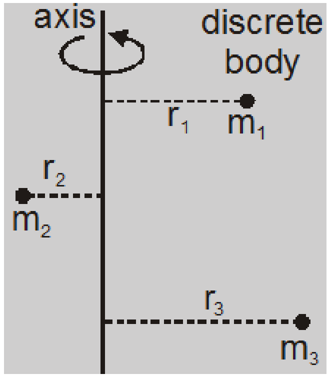
$[ML^2]$

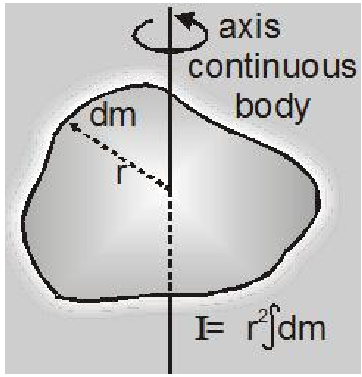


Moment of inertia of system of particle

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots..$$

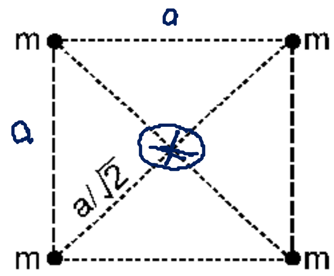
$$I = \sum mr^2$$





Example

Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.



Sol.

The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2} ma^2$. The moment of inertia of the system is, therefore,

$$4 \times \frac{1}{2} ma^2 = 2ma^2.$$

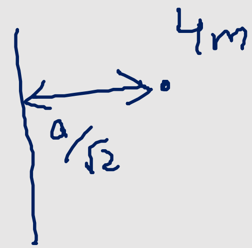
$$I = 4m\left(\frac{a}{\sqrt{2}}\right)^2$$

$$= 4 \cdot \frac{ma^2}{2}$$

$$= 2ma^2$$

$$4mK^2 = 2ma^2$$

$$K = a/\sqrt{2}$$



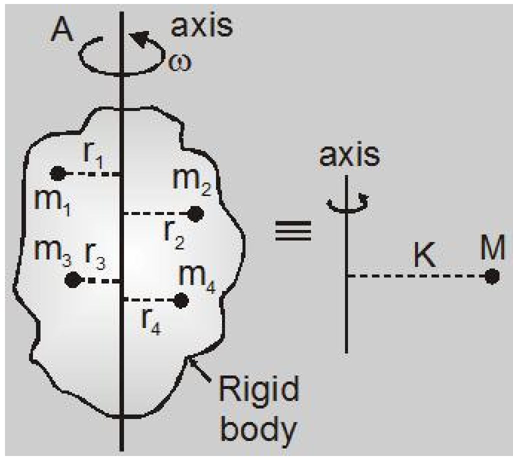
Radius of Gyration (K)

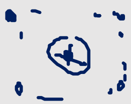
The radius of gyration of a body is the distance from axis of rotation, the square of this distance when multiplied by the mass of body then it gives the moment of inertia of the body ($I = MK^2$) about same axis of rotation.

$$I = MK^2$$

Radius of gyaration $$K = \sqrt{\frac{I}{M}}$$

where,
I = Moment of inertia of system about the axis, and
M = Total mass of the system





Theorems of moment of Inertia

Perpendicular axis theorem (applicable only for two dimensional bodies or plane laminas)

$$I_z = I_x + I_y$$

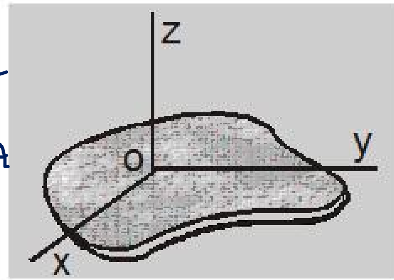
where

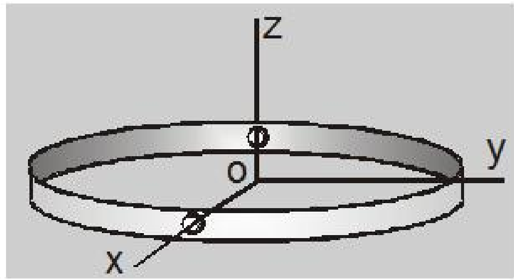
I_x = MI of the body about X-axis

I_y = MI of the body about Y-axis

(Plane of object)

I_z = MI of the body about Z-axis (⊥r to plane)





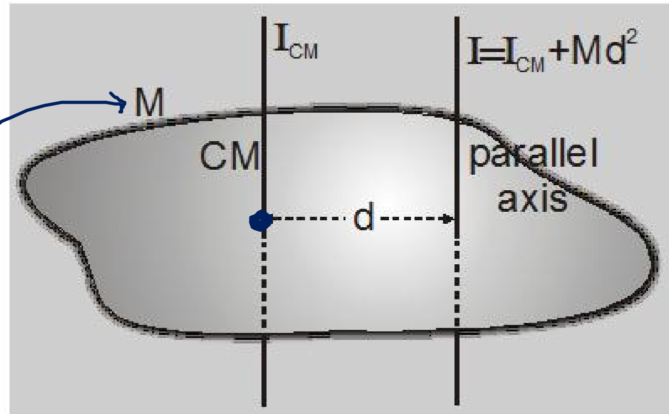
Theorems of moment of Inertia

Parallel axis theorem (applicable for all types of bodies)

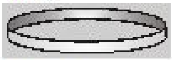
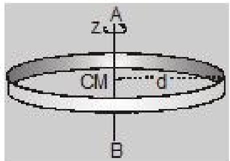
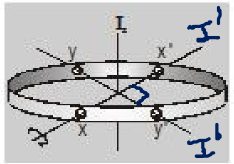
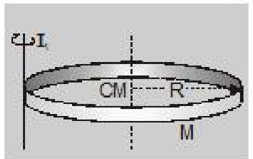
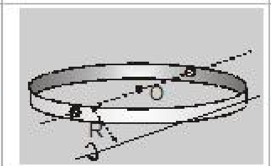
$$I = I_{CM} + Md^2$$

I_{CM} = Moment of inertia about the axis passing through centre of mass

Total mass of system



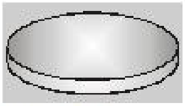
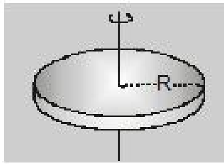
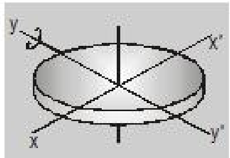
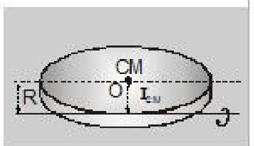
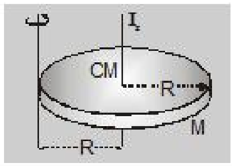
Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(1) Circular Ring  Mass = M Radius = R	(a) About an axis perpendicular to the plane and passes through the centre		MR^2	R
	(b) About the diametric axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(c) About an axis tengential to the rim and perpendicular to the plane of the ring		$2MR^2$	$\sqrt{2}\,R$
	(d) About an axis tangential to the rim and lying in the plane of ring		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}\,R$

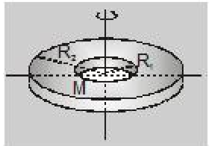
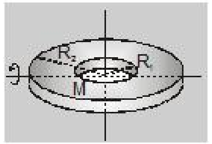
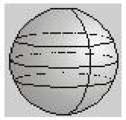
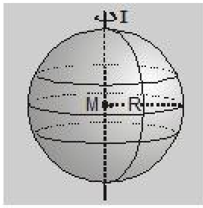
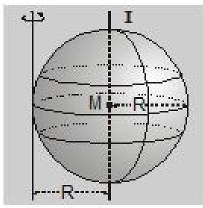
$2I' = I$

$I' = I/2 = \frac{MR^2}{2}$

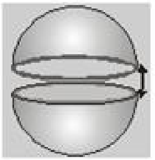
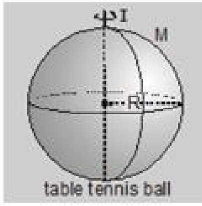
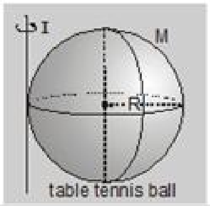
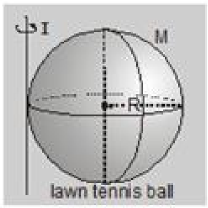
Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
(2) Circular Disc  Mass = M Radius = R	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(b) About a diametric axis		$\frac{MR^2}{4}$	$\frac{R}{2}$
	(c) About an axis tangential to the rim and lying in the plane of the disc		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$
	(d) About an axis tangential to the rim & perpendicular to the plane of disc		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$

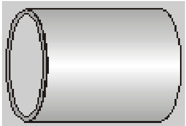
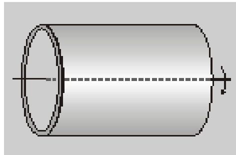
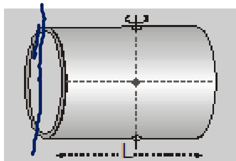
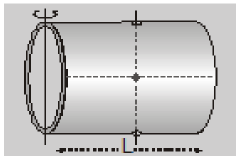
Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(3) Annular disc  M = Mass R_1 = Internal Radius R_2 = Outer Radius ($R_2 > R_1$)	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{M}{2}\left[R_1^2+R_2^2\right]$	$\sqrt{\frac{R_1^2+R_2^2}{2}}$
	(b) About a diameteric axis		$\frac{M}{4}\left[R_1^2+R_2^2\right]$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
(4) Solid Sphere  M = Mass R = Radius	(a) About its diametric axis which passes through its centre of mass		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
	(b) About a tangent to the Sphere		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$

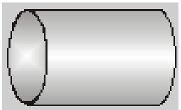
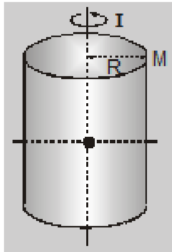
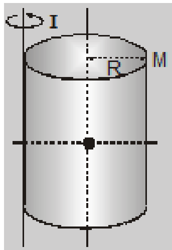
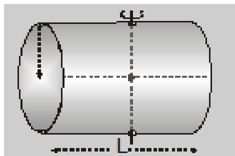
Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(5) Hollow Sphere (Thin spherical Shell)  M = Mass R = Radius Thickness negligible	(a) About diametric axis passing through centre of mass		$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
	(b) About a tangent to the surface		$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
(6) Hollow sphere with cavity  r=Internal radius R=Outer radius M=Mass	About diametric axis passes through centre of mass		$\frac{2}{5}M\frac{(R^5-r^5)}{(R^3-r^3)}$	$\sqrt{\frac{2}{5}\frac{(R^5-r^5)}{(R^3-r^3)}}$

Moment of inertia of some regular bodies

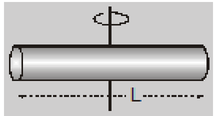
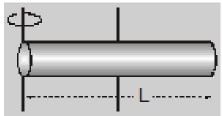
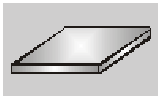
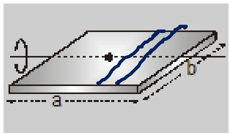
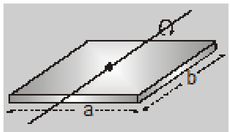
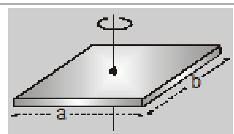
Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(7) Hollow Cylinder  M = Mass R = Radius L = Length	(a) About its geometrical axis which is parallel to its length		MR^2	R
	(b) About an axis which is perpendicular to its length and passes through its centre of mass		$M\left[\frac{R^2}{2}+\frac{L^2}{12}\right]$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{12}}$
	(c) About an axis perpendicular to its length and passing through one end of the cylinder		$M\left[\frac{R^2}{2}+\frac{L^2}{3}\right]$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{3}}$

Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(8) Solid Cylinder M = Mass R = Radius L = Length 	(a) About its geometrical axis, which is parallel to its length		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	(b) About an axis tangential to the cylinderical surface and parallel to its geometrical axis		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
	(c) About an axis passing through the centre of mass and perpendicular to its length		$M\left[\frac{L^2}{12}+\frac{R^2}{4}\right]$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{4}}$

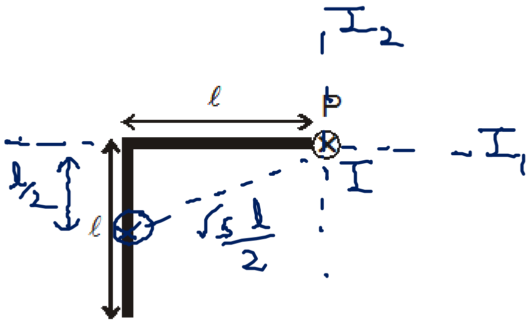
rod
ring
disc
Solid sph.
Hollow sph.
Hollow cyl.
Solid cyl.

Moment of inertia of some regular bodies

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration(K)
(9) Thin Rod 	(a) About an axis passing through centre of mass and perpendicular to its length		$\frac{ML^2}{12}$	$\frac{L}{2\sqrt{3}}$
Thickness is negligible w.r.t. length	(b) About an axis passing through one end and perpendicular to length of the rod		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
(10) Rectangular Plate 	(a) About an axis passing through centre of mass and perpendicular to side b in its plane		$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
M = Mass a = Length b = Breadth	(b) About an axis passing through centre of mass and perpendicular to side a in its plane.		$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$
	(c) About an axis passing throught centre of mass and perpendicular to plane		$\frac{M}{12}(a^2+b^2)$	$\sqrt{\frac{a^2+b^2}{12}}$

Example

Find the moment of inertia of the two uniform joint rods having mass m each about point P as shown in figure. ~~Using parallel axis theorem.~~

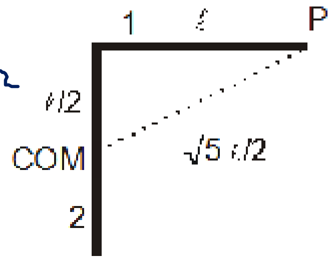


Sol.

Moment of inertia of rod 1 about axis P , $I_1 = \frac{m\ell^2}{3}$

Moment of inertia of rod 2 about axis P, $I_2 = \frac{m\ell^2}{12} + m\left(\sqrt{5}\,\frac{\ell}{2}\right)^2$

So moment of inertia of a system about axis P ,



$I = I_1 + I_2 = \frac{m\ell^2}{3} + \frac{m\ell^2}{12} + m\left(\sqrt{5}\,\frac{\ell}{2}\right)^2 = \frac{5m\ell^2}{12} + \frac{5}{4}m\ell^2$

$= \frac{5m\ell^2(1+3)}{12}$

$= \frac{5m\ell^2}{3}$

$I = \frac{5m\ell^2}{3}$

$I_1 = \frac{m\ell^2}{3}$

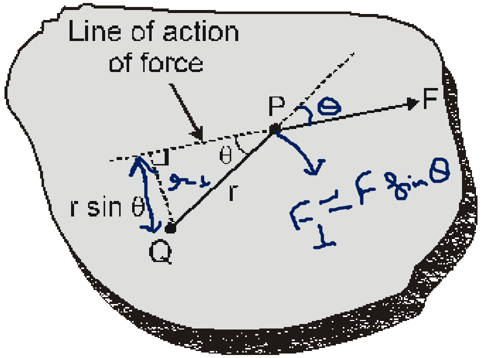
$I_2 = \frac{m\ell^2}{3} + m\ell^2 = \frac{4}{3}m\ell^2$

$I = I_1 + I_2$

$= \frac{5}{3}m\ell^2$

Torque

Torque represents the capability of a force to produce change in the rotational motion of the body.



6.1 Torque about a point :

Torque of force $\vec{F}$ about a point $\vec{\tau} = \vec{r} \times \vec{F}$

Where $\vec{F}$ = force applied

P = point of application of force

Q = Point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force w.r.t. the point about which we want to determine the torque.

$$|\vec{\tau}| = r F \sin\theta = r_{\perp} F = r F_{\perp}$$

Where θ = angle between the direction of force and the position vector of P wrt. Q.

$r_{\perp} = r \sin\theta$ = perpendicular distance of line of action of force from point Q ,it is also called force arm.

$F_{\perp} = F \sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$

SI unit of torque is Nm

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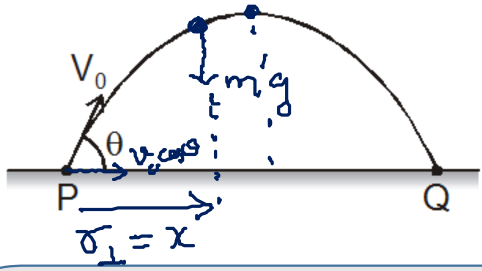
Example

A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with horizontal. Find out the torque about the point of projection acting on the particle when it is at its maximum height ?



Sol.

$$\tau = rF\sin\theta = \frac{R}{2}\,mg = \frac{v_0^2 \sin 2\theta}{2g}\,mg$$

$$\tau = \frac{mv_0^2 \sin 2\theta}{2}$$

$\tau = r_\perp F$

$= \frac{R}{2} \cdot mg$

$\tau = r_\perp F = r F_\perp$

$x = v_0 \cos\theta \cdot t = r_\perp$

$\tau = r_\perp F$

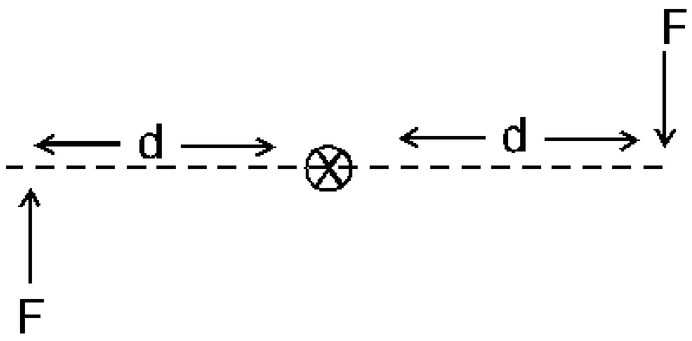
$= mg\, v_0 \cos\theta\, t$

$t = \frac{T}{2} = \frac{u \sin\theta}{g}$

$\tau = \frac{mg\, v_0 \cos\theta\, v_0 \sin\theta}{g}$

Force Couple

A pair of forces each of same magnitude and acting in opposite direction is called a force couple.
Torque due to couple = Magnitude of one force × distance between their lines of action.
Magnitude of torque = $\tau = F\ (2d)$



A couple does not exert a net force on an object even though it exerts a torque.

Net torque due to a force couple is same about any point.

Rotation about a fixed axis

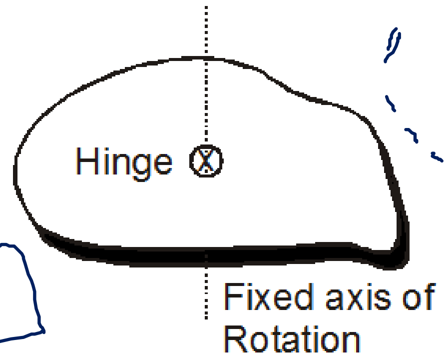
If I_{Hinge} = moment of inertia about the axis of rotation (since this axis passes through the hinge, hence the name I_{Hinge}).

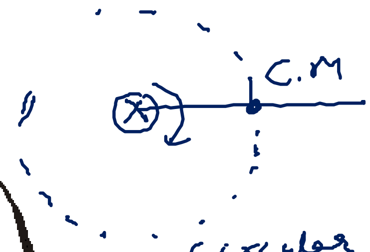
Tau → $\vec{\tau}_{ext})$ = resultant external torque acting on the body about axis of rotation

α = angular acceleration of the body.

$$\vec{\tau}_{ext})_{Hinge} = I_{Hinge}\ \vec{\alpha}$$

$$\vec{F} = M\vec{a}$$





circular

$$a_c = \omega^2 r = v^2/r$$

$$a_t = \alpha r$$

Rotational Kinetic Energy = $\frac{1}{2}.I.\omega^2$

$$W = \int \vec{F}.\vec{ds}$$

$$W = \int \vec{\tau}.\vec{d\theta}$$

$$\vec{P} = M\vec{v}_{CM}$$

$$\vec{F}_{external} = M\vec{a}_{CM}$$

Net external force acting on the body has two component tangential and centripetal.

$$\Rightarrow \quad F_c = ma_c = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM}$$

$$\Rightarrow \quad F_t = ma_t = m\alpha\, r_{CM}$$

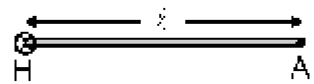
$$P = \vec{F}.\vec{v}$$

$$P = \vec{\tau}.\vec{\omega}$$

Example

A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.

(i) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest ?



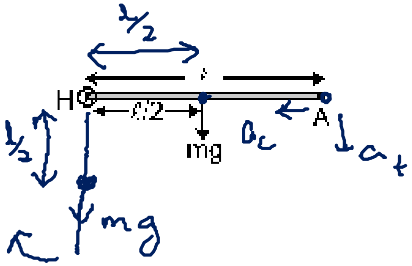
(ii) Calculate the acceleration (tangential and radial) of point A at this moment.

(iii) Find α and ω when rod becomes vertical.

$K.E. = \frac{1}{2} I\omega^2$

Sol. (i) $\tau_H = I_H \alpha$

$mg.\frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \rightarrow \alpha = \frac{3g}{2\ell}$



(ii) $a_t = \alpha\ell = \frac{3g}{2\ell}.\ell = \frac{3g}{2}$

$a_c = \omega^2 r = 0.\ell = 0$ $\quad (\because \omega = 0$ just after release)

(iii) Torque = 0 when rod becomes vertical.

so $\alpha = 0$

using energy conservation $\quad \frac{mg\ell}{2} = \frac{1}{2}I\omega^2 \quad \left(I = \frac{m\ell^2}{3}\right)$

$\omega = \sqrt{\frac{3g}{\ell}}$

Loss in P.E. = Gain in K.E

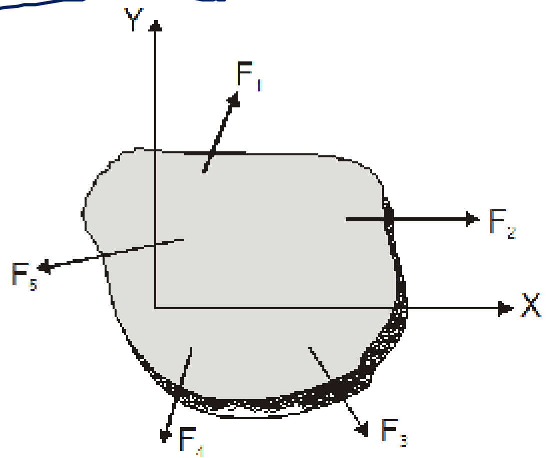
$mg\frac{\ell}{2} = \frac{1}{2}I\omega^2$

Equilibrium

A system is in mechanical equilibrium if it is in translational as well as rotational equilibrium.

For this : $\overline{F_{net}} = 0$

$\overline{\tau_{net}}$ 0 (about every point)



From (6.3), if $\overline{F_{net}}$ 0 then $\overline{\tau_{net}}$ is same about every point

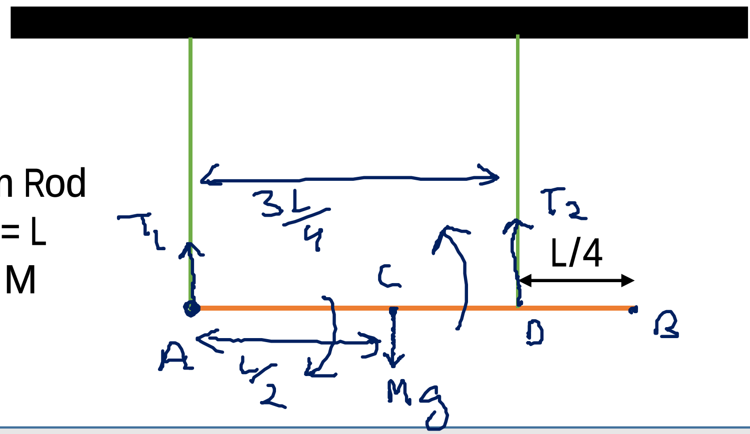
Hence necessary and sufficient condition for equilibrium is $\overline{F_{net}} = 0$, $\overline{\tau_{net}} = 0$ about any one point,

which we can choose as per our convenience. ($\overline{\tau_{net}}$ will automatically be zero about every point)

Example

In the figure shown, find tension in both the strings if rod is in equilibrium.

Uniform Rod
Length = L
Mass = M



Sol.

$\vec{F}_{net} = 0$

$T_1 + T_2 = Mg \quad -(1)$

$\vec{\tau}_{net}\Big)_A = 0$

$Mg\frac{L}{2} - T_2 \cdot \frac{3L}{4} = 0$

$$T_2 = \frac{2Mg}{3} \quad -(2)$$

$T_1 = Mg - \frac{2Mg}{3} = \frac{Mg}{3}$

$$T_1 = \frac{Mg}{3}$$