Integral as an Antiderivative

If f and g are functions of x such that $\boxed{g'(x) = f(x)}$, then the function g is called a anti-derivative (or primitive function or simply integral) of f w.r.t. x.

It is written symbolically,

$$\boxed{\frac{d}{dx}\{g(x)\} = f(x)}$$

$$\int f(x)\,dx = g(x) + C$$

$\frac{d}{dx}\sin x = \cos x$.

$\int \cos x\,dx = \sin x + C$

$\sin x + 1$

$\sin x + 5$

$\sin x + 100$

- So $\int f(x)dx$ is not unique due to presence of 'c'.
- Derivative of a function is unique, but integration is not unique

STANDARD RESULTS :

(i) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,\ n \neq -1 \quad \Rightarrow \quad \boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1}$

(ii) $\frac{d}{dx}(\log|x|) = \frac{1}{x} \quad \Rightarrow \quad \int \frac{1}{x}\,dx = \log|x| + C$, when $x \neq 0$

(iii) $\frac{d}{dx}(e^x) = e^x \quad \Rightarrow \quad \int e^x\,dx = e^x + C$

(iv) $\frac{d}{dx}\left(\frac{a^x}{\log_e a}\right) = a^x,\ a > 0,\ a \neq 1 \quad \Rightarrow \quad \int a^x\,dx = \frac{a^x}{\log a} + C$

(v) $\frac{d}{dx}(-\cos x) = \sin x \quad \Rightarrow \quad \int \sin x\,dx = -\cos x + C$

(vi) $\frac{d}{dx}(\sin x) = \cos x \quad \Rightarrow \quad \int \cos x\,dx = \sin x + C$

(vii) $\frac{d}{dx}(\tan x) = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x\,dx = \tan x + C$

(viii) $\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x \quad \Rightarrow \quad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$

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STANDARD RESULTS :

(ix) $\frac{d}{dx}(\sec x) = \sec x \tan x \Rightarrow \int \sec x \tan x \, dx = \sec x + C$

(x) $\frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x \Rightarrow \int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$

(xi) $\frac{d}{dx}(\log|\sin x|) = \cot x \Rightarrow \int \cot x \, dx = \log|\sin x| + C$

(xii) $\frac{d}{dx}(-\log|\cos x|) = \tan x \Rightarrow \int \tan x \, dx = -\log|\cos x| + C$

(xiii) $\frac{d}{dx}(\log|\sec x + \tan x|) = \sec x \Rightarrow \int \sec x \, dx = \log|\sec x + \tan x| + C$

(xiv) $\frac{d}{dx}(\log|\operatorname{cosec} x - \cot x|) = \operatorname{cosec} x$

$$\Rightarrow \int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + C$$

Techniques of Integration

- Substitution
- By part (product rule)
- Partial (fraction)
- Misc.

$x = \phi(t) \Rightarrow dx = \phi'(t)\,dt.$

$$\boxed{\int f(x)\,dx = \int f(\phi(t))\,\phi'(t)\,dt.}$$

Method of Substitution : (Change of independent variable)

Integration can be made easier with the help of substitution. Here x is substituted by $\phi(t)$, where $\phi(t)$ is a continuously differentiable function and we have

$$\int f(x)dx = \int f\{\phi(t)\}.\phi'(t)dt \text{ . where } x = \phi(t)$$

There is no general rule for finding a proper substitution and the best guide in this matter is experience.

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$$\int \frac{\cos(\ln x)}{x} dx$$

$\ln x = t$

$\frac{1}{x} dx = dt$

$\Rightarrow \int \cos t \, dt = \sin t$

$\boxed{\sin(\ln x) + c}$

$$\int \frac{x^3 dx}{1+x^8}$$

Put $x^4 = t$

$4x^3 \, dx = dt$

$\boxed{x^3 \, dx = \frac{dt}{4}}$

$\Rightarrow \int \frac{dt/4}{1+t^2} = \frac{1}{4} \int \frac{1}{1+t^2} dt = \frac{1}{4} \tan^{-1} t = \frac{1}{4} \tan^{-1} x^4 + c.$

- $\sin(\ln x) + c$ ✓
- ✓ $\frac{1}{4} \tan^{-1}(x^4) + c$

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$$\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$$

$\rightarrow \tan^{-1} x^3 = t$

$\frac{1}{1+x^6} \times 3x^2\, dx = dt \Rightarrow \frac{x^2}{1+x^6} dx = \frac{dt}{3}$

$\Rightarrow \int \frac{t\, dt}{3} = \frac{1}{3}\frac{t^2}{2} + C = \frac{t^2}{6} = \frac{(\tan^{-1} x^3)^2}{6} + C$

$$\int \frac{\cos 2x}{\sin x} dx$$

$\Rightarrow \int \frac{1-2\sin^2 x}{\sin x} dx = \int (\operatorname{cosec} x - 2\sin x)\, dx$

$= \int \operatorname{cosec} x - 2\int \sin x\, dx$

$= \ln(\operatorname{cosec} x - \cot x) + 2\cos x + C.$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 2\cos^2 x - 1$

$= 1 - 2\sin^2 x.$

$= \frac{1-\tan^2 x}{1+\tan^2 x}$

- $\frac{1}{6}(\tan^{-1} x^3)^2 + c$ ✓

- ✓ $\ln(\operatorname{cosec} x - \cot x) + 2\cos x + c$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c \;;$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c\;;$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c \;;$$

*

$$\boxed{\int \frac{dx}{\sqrt{x^2 - a^2}} = ln\left(x + \sqrt{x^2 - a^2}\right) + c\;;}$$

*

$$\boxed{\int \frac{dx}{\sqrt{x^2 + a^2}} = ln\left(x + \sqrt{x^2 + a^2}\right) + c}$$

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$$\int \frac{\sin 2x}{\sqrt{9-\sin^4 x}}\,dx$$

1) Substitute $\sin^2 x = t$

$2\sin x \cdot \cos x \, dx = dt \Rightarrow \sin 2x \, dx = dt$

$$\int \frac{dt}{\sqrt{3^2-t^2}} = \sin^{-1}\left(\frac{t}{3}\right) + c = \sin^{-1}\left(\frac{\sin^2 x}{3}\right) + c.$$

$$\int \frac{e^x dx}{\sqrt{e^{2x}-1}}$$

2) $e^x = t \qquad e^x dx = dt$

$$\int \frac{dt}{\sqrt{t^2-1}} = \ln\left(t + \sqrt{t^2-1^2}\right) + c.$$

$$= \boxed{\ln\left(e^x + \sqrt{e^{2x}-1}\right) + c}$$

✓ $\sin^{-1}\left(\frac{\sin^2 x}{3}\right) + c$

$\ln\left(e^x + \sqrt{e^{2x}-1}\right) + c$

Standard Results

For Integration of type $\int \frac{dx}{ax^2+bx+c}$ and $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ make ax^2+bx+c as perfect square.

For Integration of type $\int \frac{px+q}{ax^2+bx+c}dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ write $px+q=\lambda(2ax+b)+\mu$

$\lambda \frac{d}{dx}(ax^2+bx+c)+\mu$

$\int \frac{dx}{\sqrt{Q}}$. $Q =$ perfect square.

$$\int \frac{e^x\,dx}{\sqrt{5-4e^x+e^{2x}}}$$

$5-4t+t^2 = t^2-4t+5$

$= \underbrace{t^2-4t+4}+1$

$= (t-2)^2+1$

$\rightarrow \quad e^x = t \Rightarrow e^x dx = dt$

$\int \frac{dt}{\sqrt{5-4t+t^2}} = \int \frac{dt}{\sqrt{1+(t-2)^2}}$ $\quad t-2=z$

$dt = dz$

$\int \frac{dz}{\sqrt{1+z^2}} = \ln\left(z+\sqrt{z^2+1}\right) = \ln\left(t-2+\sqrt{(t-2)^2+1}\right)$

$= \ln\left(e^x-2+\sqrt{5-4e^x+e^{2x}}\right) + C.$

Ans.

$\int \frac{L}{\sqrt{Q}}$ $\qquad L = \lambda \frac{d}{dx} Q + \mu$.

$$\int \frac{5x+4}{\sqrt{x^2+2x+5}}\,dx.$$

$\frac{d}{dx}(x^2+2x+5) = 2x+2$

$5x+4 = \lambda(2x+2) + \mu$.

$5 = 2\lambda$, $4 = 2\lambda + \mu$.

$\lambda = \frac{5}{2}$, $\mu = -1$.

$5x+4 = \frac{5}{2}(2x+2) - 1$

$= 5(x+1) - 1$

$\int \frac{\frac{5}{2}(2x+2)\,dx}{\sqrt{x^2+2x+5}} - \int \frac{1}{\sqrt{x^2+2x+5}}\,dx$

as a perfect square.

$x^2+2x+5 = t$

$(2x+2)\,dx = dt$.

$\int \frac{5}{2} \frac{dt}{\sqrt{t}}$

$\frac{5}{2} \frac{t^{-1/2+1}}{-\frac{1}{2}+1} = 5\sqrt{t} = 5\sqrt{x^2+2x+5}$.

$\ln\left(x+1+\sqrt{x^2+2x+5}\right) + C$.

Ans: $5\sqrt{x^2+2x+5} - \ln\left(x+1+\sqrt{x^2+2x+5}\right) + C$

INTEGRATION BY PARTS :

Theory: If f (x) and g (x) are derivable functions then

$$\frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x).f'(x)$$

$$\therefore \quad \int \underset{\mathrm{I}}{f(x)}.\underset{\mathrm{II}}{g'(x)}\, dx = f(x).g(x) - \int g(x).f'(x)\, dx$$

$$\boxed{I = \int \underbrace{f(x)}_{\mathrm{I}} \cdot \underbrace{g(x)}_{\mathrm{II}} dx}$$

$$= 1^{st} \text{ function} \times \text{integral of } 2^{nd} - \int (\text{diff. co-eff. of } 1^{st}) \times (\text{integral of } 2^{nd})\, dx$$

Remember **ILATE** for deciding the choice of the first and second function which is not arbitrary.

Here I for inverse trigonometric function
L for Logarithmic function
A for Algebraic function
T for Trigonometric function
E for Exponential Function

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ILATE

$$\int \underset{\text{II}}{x}\, \underset{\text{I}}{\tan^{-1} x}\, dx$$

$$\int (\tan^{-1}x)\, x\, dx = \tan^{-1}x \int x\, dx - \int \left(\frac{d}{dx}\tan^{-1}x \int x\, dx \right) dx$$

$$= \tan^{-1}x \times \frac{x^2}{2} - \int \frac{1}{1+x^2} \times \frac{x^2}{2}\, dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2}\, dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2} \right) dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2} \left(x - \tan^{-1}x \right) + C$$

Two Classic Integrands :

(a) * $\int e^x(f(x)+f'(x))dx = e^x f(x)+C$ & (b) $\int (f(x)+xf'(x))dx = x\ f(x) + C$

Proof :

(a) $\int e^x(f(x)+f'(x))dx = \int e^x f(x)dx + \int \underset{I}{e^x} \underset{II}{f'(x)} dx$

$= \int e^x f(x)dx + e^x f(x) - \int e^x f(x)\, dx + c = e^x f(x) + c$

(b) $\int f(x)\, dx + x\, f'(x)dx = \int f(x)\, dx + \int \underset{I}{x} \underset{II}{f'(x)}\, dx$

$= \int f(x)dx + x\, f(x) - \int f(x)dx = x\ f(x) + c$

$f(x) + f'(x)$

$e^x(\sin x + \cos x)dx$

$= e^x \sin x + C$

$\int(f(x) + x f'(x))dx$

$= x f(x) + C.$

$$\frac{x^2}{1+x^2} = \cancel{1+x^2-x^2} = \frac{1+x^2-1}{1+x^2} = 1-\frac{1}{1+x^2}$$

$$\int \frac{xe^x}{(1+x)^2}\,dx = \frac{(1+x-1)\,e^x}{(1+x)^2} = e^x\left[\underbrace{\frac{1}{1+x}}_{f(x)} - \underbrace{\frac{1}{(1+x)^2}}_{f'(x)}\right]dx$$

$$= \int e^x\,(f(x)+f'(x))\,dx$$

$$= e^x f(x) + C$$

$$= e^x \times \frac{1}{1+x} + C$$

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$$\int (\sin x + x\cos x)dx = x\sin x + c$$

$$\int (f(x) + x f'(x)) dx = x f(x) + c$$

INTEGRALS OF TRIGONOMETRIC FUNCTIONS :

Type – 1 : $\int \frac{dx}{a+b\sin^2 x}$ / $\int \frac{dx}{a+b\cos^2 x}$ / $\int \frac{dx}{a\sin^2 x+b\cos^2 x+c\sin x\cos x}$ / $\frac{dx}{(a\cos x+b\sin x)^2}$

Multiply N^r and D^r by $\sec^2 x$ or $\operatorname{cosec}^2 x$ and proceed

Type – 2 : $\int \frac{dx}{a+b\sin x}$ / $\int \frac{dx}{a+b\cos x}$ / $\int \frac{dx}{a+b\sin x+c\cos x}$

Convert sin x and cos x into their corresponding tangent to half the angles and put $\tan\frac{x}{2}=t$

$A(D_r) + B\left(\frac{d}{dx}D_r\right) + C$

Type–3 : $\int \frac{a\sin x+b\cos x+c}{\ell\sin x+m\cos x+n}dx$; $\boxed{N^r = A(D^r)+B\left(\frac{d}{dx}D^r\right)+C}$

Type–4 : $\int \frac{x^2+1}{x^4+kx^2+1}dx$ **or** $\int \frac{x^2-1}{x^4+kx^2+1}dx$

Divide N^r and D^r by x^2 and take suitable substituion

$$\int \frac{dx}{(3\sin x - 4\cos x)^2}$$

Multiplied $\sec^2 x$ in Nr & Dr

$$\int \frac{\sec^2 x\, dx}{(3\sin x.\sec x - 4\cos x.\sec x)^2} = \int \frac{\sec^2 x\, dx}{(3\tan x - 4)^2}$$

$$3\tan x - 4 = t$$

$$3\sec^2 x\, dx = dt$$

$$\sec^2 x\, dx = dt/3$$

$$\int \frac{dt}{3t^2} = \frac{1}{3}\int t^{-2}\, dt$$

$$= \frac{1}{3}\frac{t^{-2+1}}{-2+1} = -\frac{1}{3t} + C$$

$$= \boxed{-\frac{1}{3(3\tan x - 4)} + C}$$

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$$\int \frac{dx}{5+4\cos x}$$

$$\frac{dx}{5+4\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)} = \int \frac{\sec^2 x/2 \, dx}{5+5\tan^2\frac{x}{2}+4-4\tan^2\frac{x}{2}}$$

$$= \int \frac{\sec^2 x/2 \, dx}{9+\tan^2 x/2}$$

put $\tan\frac{x}{2} = t$

$\sec^2\frac{x}{2} \times \frac{1}{2} dx = dt$

$$\int \frac{2t}{9+t^2} = \frac{2}{3}\tan^{-1}\frac{t}{3}$$

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$$\int \frac{x^2+1}{x^4+7x^2+1}\,dx$$

Divide Nr & Dr by x^2

$$\int \frac{1+\frac{1}{x^2}}{x^2+7+\frac{1}{x^2}}\,dx$$

$$\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x^2+\frac{1}{x^2}-2\right)+9} = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3^2}$$

$x-\frac{1}{x}=t$

$\left(1+\frac{1}{x^2}\right)dx=dt$

$$\int \frac{dt}{t^2+3^2} = \frac{1}{3}\tan^{-1}\frac{t}{3}$$

$$\int \frac{x^2+1}{x^4+7x^2+1}dx$$

Divide Nr & Dr by x^2

$$\int \frac{1+\frac{1}{x^2}}{x^2+7+\frac{1}{x^2}}dx$$

$$\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x^2+\frac{1}{x^2}-2\right)+9} = \int \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+3^2}$$

$x-\frac{1}{x}=t$

$\left(1+\frac{1}{x^2}\right)dx=dt$

$$\int \frac{dt}{t^2+3^2} = \frac{1}{3}\tan^{-1}\frac{t}{3}$$