Problems Solving on Differential Calculus

By
Ankush Garg(B. Tech, IIT Jodhpur)

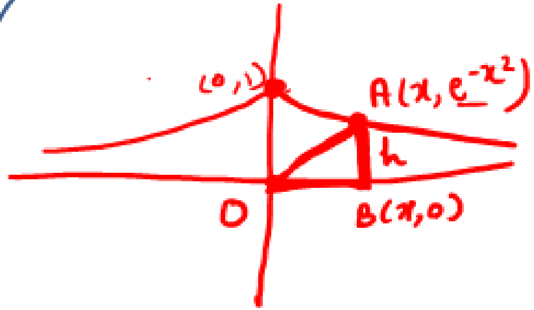
Q. Point 'A' lies on the curve $y = e^{-x^2}$ and has the coordinate (x, e^{-x^2}) where $x > 0$. Point B has the coordinates $(x, 0)$. If 'O' is the origin then the maximum area of the triangle AOB is.

(A) $\dfrac{1}{\sqrt{2e}}$

(B) $\dfrac{1}{\sqrt{4e}}$

(C) $\dfrac{1}{\sqrt{e}}$

(D) $\dfrac{1}{\sqrt{8e}}$



$A = \frac{1}{2} \times B \times H$

$A = \frac{1}{2} \times x \times e^{-x^2}$

$\frac{dA}{dx} = \frac{1}{2}\left[e^{-x^2} + x e^{-x^2}(-2x)\right]$

$0 = \frac{1}{2} e^{-x^2}\left[1 - 2x^2\right]$

$1 - 2x^2 = 0$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

$x = \frac{1}{\sqrt{2}}$ ✓

$A = \frac{1}{2} \times \frac{1}{\sqrt{2}} \times e^{-1/2}$

$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{e}}$

$\boxed{A = \frac{1}{\sqrt{8e}}}$

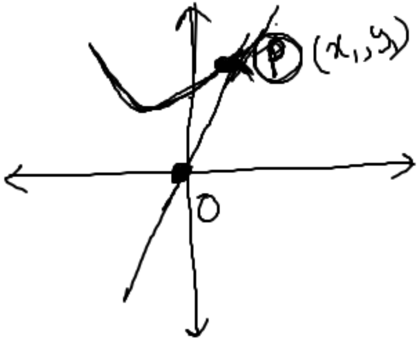
Q. The coordinates of the point on the curve $y = x^2 + 3x + 4$ the tangent at which passes through the origin is equal to

(A) (2, 14), (–2, 2)

(B) (2, 14), (–2, –2)

(C) (2, 14), (2, 2)

(D) None of these



$$y = x^2 + 3x + 4$$

$$\left.\frac{dy}{dx}\right|_{(x_1, y_1)} = 2x_1 + 3$$

Eqn of Tangent

$$y - y_1 = (2x_1 + 3)(x - x_1)$$

(0,0)

$$0 - y_1 = (2x_1 + 3)(-x_1)$$

$$-[x_1^2 + 3x_1 + 4] = -2x_1^2 - 3x_1$$

$$x_1^2 - 4 = 0$$

$$\boxed{x_1 = \pm 2}$$

$$P(2, 2^2 + 3\times2 + 4)$$

$$P(2, 14), \quad Q(-2, (-2)^2 + 3(-2) + 4)$$

$$Q(-2, 2)$$

Q. If f and g are two increasing functions such that gof is defined, then

(A) gof is an increasing function

(B) gof is a decreasing function

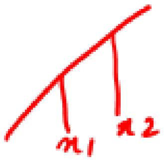
(C) gof is neither increasing nor decreasing

(D) None of these

$f \uparrow$ $\quad$ $g \uparrow$

$x_2 > x_1$

$g(x_2) > g(x_1)$



$x_2 > x_1$

$f(x_2) > f(x_1)$ [f is $\uparrow$]

$g(f(x_2)) > g(f(x_1))$ [g is $\uparrow$]

$gof(x_2) > gof(x_1)$ $\uparrow$

Q. The equation $\sin x + x\cos x = 0$ has at least one root in

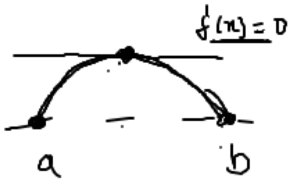
Rolle's

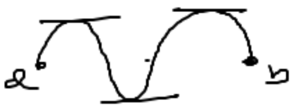
(A) $\left(-\dfrac{\pi}{2}, 0\right)$

(B) $(0, \pi)$

(C) $\left(\pi, \dfrac{3\pi}{2}\right)$

(D) $\left(0, \dfrac{\pi}{2}\right)$





$\boxed{\sin x + x\cos x} = 0$

$\dfrac{d}{dx}(x\sin x)$

$\boxed{f(x) = x\sin x}$

① continuous ✓

② diff ✓

$0 = f'(x)$

$f(a) = f(b)$

$f(a) = f(b)$

$[a, b]$

$\sin x + x\cos x = 0$

$x\sin x$

Q. Let $\ell = \lim\limits_{x\to 2} \dfrac{[x]}{\{x\}}$ and $m = \lim\limits_{x\to 2} \dfrac{\{x\}}{[x]}$

Where [y] and {x} denotes largest integer less than or equal to y and frac... y respectively.

(A) l exist but m does not

(B) m exist but l does not

(C) Both l and m exist

(D) Neither l nor m exist

$\{x\}$, $[x]$

$\boxed{R.H.L = L.H.L = \text{finite}}$

$\boxed{\{x\} = x - [x]}$

$l = \lim\limits_{x\to 2} \dfrac{[x]}{\{x\}}$

$\underline{L.H.L}$ $\lim\limits_{x\to 2^-} \dfrac{1}{x-[x]}$

$\lim\limits_{x\to 2} \dfrac{1}{x-1}$

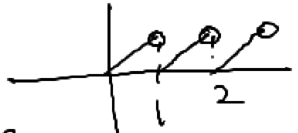
$\boxed{L.H.L = 1}$

$R.H.L = \lim\limits_{x\to 2^+} \dfrac{[x]}{x-[x]}$

$= \lim\limits_{x\to 2^+} \dfrac{2}{x-2}$

$R.H.L = \infty$

$\boxed{l \text{ not exist}}$



For m

$\boxed{\begin{array}{l} L.H.L = 1 \\ R.H.L = 0 \end{array}}$

m does not exist

Q. Let $\ell = \lim\limits_{x\to 2} \dfrac{[x]}{\{x\}}$ and $m = \lim\limits_{x\to 2} \dfrac{\{x\}}{[x]}$

Where [y] and {x} denotes largest integer less than or equal to y and fractional part of y respectively.

(A) l exist but m does not

(B) m exist but l does not

(C) Both l and m exist

(D) Neither l nor m exist

$\{x\}$, $[x]$

$\boxed{R{\cdot}H{\cdot}L = L{\cdot}H{\cdot}L = \text{finite}}$

$\boxed{\{x\} = x - [x]}$

$l = \lim\limits_{x\to 2} \dfrac{[x]}{\{x\}}$

L·H·L $\lim\limits_{x\to 2^-} \dfrac{1}{x-[x]}$

$\lim\limits_{x\to 2^-} \dfrac{1}{x-1}$

$\boxed{LHL = 1}$

$R{\cdot}H{\cdot}L = \lim\limits_{x\to 2^+} \dfrac{[x]}{x-[x]}$

$= \lim\limits_{x\to 2^+} \dfrac{2}{x-2}$

$R{\cdot}H{\cdot}L = \infty$

$\boxed{l \text{ not exist}}$

For m

$\boxed{\begin{array}{l} L{\cdot}H{\cdot}L = 1 \\ R{\cdot}H{\cdot}L = 0 \end{array}}$

m does not exist

SAFALTA.COM

An Initiative by अमर उजाला

Q. Let $f(x) = \frac{x - [x]}{1 - [x] + x}$, then range of f(x) is ([.] = G. I. F.)

(A) [0, 1] (B) [0, 1/2] (C) [1/2, 1] (D) [0,1/2)

$f(x) = \frac{x - [x]}{1 - [x] + x}$

$\boxed{\{x\} = x - [x]}$ $f(x) = \frac{\{x\}}{1 + \{x\}}$

$f(x) = \frac{\{x\} + 1 - 1}{1 + \{x\}}$

$f(x) = 1 - \frac{1}{1 + \{x\}}$

$0 \le \{x\} < 1$

$1 \le 1 + \{x\} < 2$

$\frac{1}{2} < \frac{1}{1 + \{x\}} \le 1$

$-1 \le \frac{-1}{1 + \{x\}} < \frac{-1}{2}$

$0 \le 1 - \frac{1}{1 + \{x\}} < \frac{1}{2}$

Q. Function $f : R \rightarrow R$, $f(x) = x^3 - x$ is.

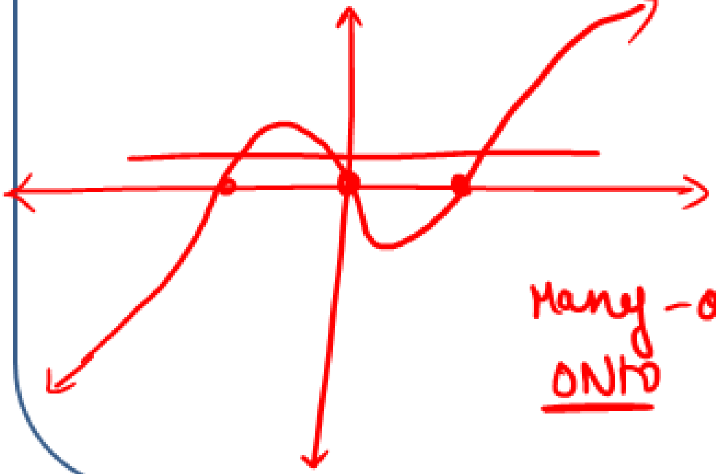
(A) One-one onto (B) One-one into ~~(C) M~~any-one onto (D) Many-one into

$f(x) = x^3 - x$

$f(x) = x(x^2 - 1)$

$f(x) = x(x-1)(x+1)$

↳ odd degree Polynomial f^n will have Range = R.



Many-one

ONTO

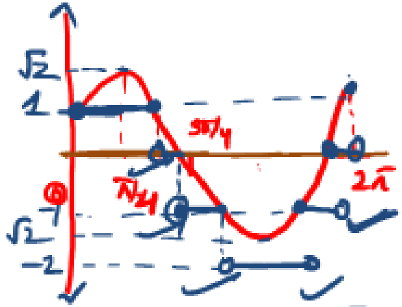
The number of points where $f(x) = [\sin x + \cos x]$ (where [ ] denotes the greatest integer function), $x \in (0, 2\pi)$ is not continuous is :

(A) 3 (B) 4 (C) 5 (D) 6

$f(x) = [\sin x + \cos x]$

$f(x) = \left[\sqrt{2}\left(\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}}\right)\right]$

$= \left[\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)\right]$



$(0, 2\pi)$

5

Q. Evaluate $\lim\limits_{x\to-\infty} x\left(x+\sqrt{1+x^2}\right)$

(A) ∞

(B)

(C)

(D)

$$\lim_{x\to-\infty} x\left(x+\sqrt{1+x^2}\right)$$

$$\lim_{x\to-\infty} x\left(x+\sqrt{1+x^2}\right)\times\frac{\left(x-\sqrt{1+x^2}\right)}{\left(x-\sqrt{1+x^2}\right)}$$

$$\lim_{x\to-\infty}\frac{x\left[x^2-(1+x^2)\right]}{x-\sqrt{1+x^2}}$$

$$\lim_{x\to-\infty}\frac{-x}{x-\sqrt{1+x^2}}=\lim_{x\to-\infty}\frac{-x}{x-\cancel{x}\sqrt{\frac{1}{x^2}+1}}$$

$$=\lim_{x\to-\infty}\frac{-x}{x+x\sqrt{\frac{1}{x^2}+1}}$$

$$\lim_{x\to-\infty}\frac{-1}{1+\sqrt{\frac{1}{x^2}+1}}$$

$$=-\frac{1}{2}$$