Derivatives

(Method of Differentiation)

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First Principle of Derivative

The derivative can be defined for $f(x)$ at any point x as

$$\frac{dy}{dx} = f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right].$$

If the function is given as $y = f(x)$ then its derivative is written as $\frac{dy}{dx} = f'(x)$.

$$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x)$$

STANDARD DERIVATIVES :

(i) $\frac{d}{dx} x^n = nx^{n-1}, x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

$\ln x = \log_e x$

(iii) $\frac{d}{dx}(a^x) = a^x \ln a$

(iv) $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e$

$\log_a x = \frac{\log_e x}{\log_e a}$

$f'(x) = \frac{1}{\log_e a} \frac{1}{x}$

$= \log_a e \times \frac{1}{x}$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(x) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(xi) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

(i) $\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{(1-x^2)}}$

(ii) $\dfrac{d}{dx}(\cos^{-1}x) = \dfrac{-1}{\sqrt{(1-x^2)}}$

(iii) $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1+x^2}$

(iv) $\dfrac{d}{dx}(\cot^{-1}x) = \dfrac{-1}{1+x^2}$

(v) $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{|x|\sqrt{x^2-1}}$

(vi) $\dfrac{d}{dx}(\operatorname{cosec}^{-1}x) = \dfrac{-1}{|x|\sqrt{x^2-1}}$

T-1 : $\frac{d}{dx}(f_1(x) \pm f_2(x)) = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)$.

$\frac{d}{dx}(f_1(x) \pm f_2(x)) = \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x)$

T-2 : $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x)$, where k is any constant.

$\frac{d}{dx}K(f(x)) = K\frac{d}{dx}f(x)$

T-3 : PRODUCT RULE :

$$\frac{d}{dx}\{f_1(x)\, f_2(x)\} = f_1(x)\frac{d}{dx}f_2(x) + f_2(x)\frac{d}{dx}f_1(x).$$

Note : If 3 functions are involved then remember

$$D(f(x)\cdot g(x)\cdot h(x)) = f(x)\cdot g(x)\cdot h'(x) + g(x)\cdot h(x)\cdot f'(x) + h(x)\cdot f(x)\cdot g'(x)$$

T-4 QUOTIENT RULE :

$$y=\frac{f(x)}{g(x)}$$ ✓

$$D\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{g^2(x)}$$, to be remembered as

$$D\left(\frac{N^r}{D^r}\right)=\frac{D^r\frac{d}{dx}(N^r)-N^r\frac{d}{dx}(D^r)}{(D^r)^2};$$

T-5 DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE)

If $y = f(u)$ & $u = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ " **Chain Rule** "

The derivative of a composite function can also be expressed as follows. $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is defined then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

$y = f(g(x))$

$$\boxed{\frac{dy}{dx} = f'(g(x)) \times g'(x)}$$

$\frac{d}{dx} \sin(e^x) = \cos(e^x) \times e^x$

$\frac{d}{dx} \sin \log x^2 = \cos(\log x^2) \times \frac{1}{x^2} \times 2x$

T-5 DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE)

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n\,[f(x)]^{n-1} \cdot f'(x)$
$\sqrt{f(x)}$	$\frac{f'(x)}{2\sqrt{f(x)}}$
$\frac{1}{[f(x)]^n}$	$-\frac{n \cdot f'(x)}{[f(x)]^{n+1}}$
$\sin\,[f(x)]$	$\cos\,[f(x)] \cdot f'(x)$
$\cos\,[f(x)]$	$-\sin\,[f(x)] \cdot f'(x)$
$\tan\,[f(x)]$	$\sec^2\,[f(x)] \cdot f'(x)$
$\sec\,[f(x)]$	$\sec\,[f(x)] \cdot \tan\,[f(x)] \cdot f'(x)$

y	$\frac{dy}{dx}$
$\cot\,[f(x)]$	$-\operatorname{cosec}^2\,[f(x)] \cdot f'(x)$
$\operatorname{cosec}\,[f(x)]$	$-\operatorname{cosec}\,[f(x)] \cdot \cot\,[f(x)] \cdot f'(x)$
$a^{f(x)}$	$a^{f(x)} \cdot \log a \cdot f'(x)$
$e^{f(x)}$	$e^{f(x)} \cdot f'(x)$
$\log\,[f(x)]$	$\frac{f'(x)}{f(x)}$
$\log_a\,[f(x)]$	$\frac{f'(x)}{f(x)\log a}$

Theorem on Derivatives

T-5 DIFFERENTIATION OF COMPOSITE FUNCTION (CHAIN RULE)

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n\,[f(x)]^{n-1} \cdot f'(x)$
$\sqrt{f(x)}$	$\dfrac{f'(x)}{2\sqrt{f(x)}}$
$\dfrac{1}{[f(x)]^n}$	$-\dfrac{n \cdot f'(x)}{[f(x)]^{n+1}}$
$\sin[f(x)]$	$\cos[f(x)] \cdot f'(x)$
$\cos[f(x)]$	$-\sin[f(x)] \cdot f'(x)$
$\tan[f(x)]$	$\sec^2[f(x)] \cdot f'(x)$
$\sec[f(x)]$	$\sec[f(x)] \cdot \tan[f(x)] \cdot f'(x)$

y	$\frac{dy}{dx}$
$\cot[f(x)]$	$-\operatorname{cosec}^2[f(x)] \cdot f'(x)$
$\operatorname{cosec}[f(x)]$	$-\operatorname{cosec}[f(x)]$ [illegible]
$a^{f(x)}$	$a^{f(x)} \cdot$ [illegible]
$e^{f(x)}$	[illegible]
$\log[f(x)]$	[illegible]
$\log_a[f(x)]$	[illegible]

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If $y = \dfrac{\sec x - \tan x}{\sec x + \tan x}$, then $\dfrac{dy}{dx}$ equals-

(A) $2 \sec x (\sec x - \tan x)^2$

(B) $-2 \sec x (\sec x - \tan x)^2$

(C) $2 \sec x (\sec x + \tan x)^2$

(D) $-2 \sec x (\sec x + \tan x)^2$

$$y = \frac{(\sec x - \tan x)}{(\sec x + \tan x)} \frac{(\sec x - \tan x)}{(\sec x - \tan x)} = \frac{(\sec x - \tan x)^2}{\sec^2 x - \tan^2 x} = (\sec x - \tan x)^2 =$$

$$\frac{dy}{dx} = 2(\sec x - \tan x)\left[\sec x \tan x - \sec^2 x\right] = 2 \sec x (\sec x - \tan x)(\tan x - \sec x)$$

$$= -2 \sec x (\sec x - \tan x)^2$$

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If $y = \log_x 10$, then the value of dy/dx equals-

(A) 1/x (B) 10/x (C) $-\frac{(\log_x 10)^2}{x \log_e 10}$ (D) $\frac{1}{(x \log_e 10)}$

$y = \log_x 10 = \frac{\log_e 10}{\log_e x} = (\ln 10)\,\frac{1}{(\ln x)}$

$\boxed{\log_b a = \frac{1}{\log_a b}. \qquad \log_{1/b} a = \frac{\log_c a}{\log_c b}}$

$\frac{dy}{dx} = (\ln 10)\left(\frac{-1}{(\ln x)^2}\right) \times \frac{1}{x}.$

$\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$

$= -\frac{\log_e 10}{x(\log_e x)^2} = -\frac{\log_e 10}{x} \times \frac{(\log_{10} e)^2}{(\log_{10} x)^2}$

$= -\frac{(\log_{10} e)(\log_x 10)^2}{x} = \frac{-(\log_x 10)^2}{x \log_e 10}$

Differentiation by using Trigonometric Transformations :

With the help of trigonometric transformations, the labour involved in computing derivative can be reduced. Note some standard trigonometric substitutions

1. $\sqrt{a^2 - x^2} \Rightarrow$ put $x = a\sin\theta$ or $a\cos\theta$ $\sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta}$

2. $\sqrt{a^2 + x^2} \Rightarrow$ put $x = a\tan\theta$ or $a\cot\theta$

3. $\sqrt{x^2 - a^2} \Rightarrow$ put $x = a\sec\theta$ or $a\operatorname{cosec}\theta$

4. $\sqrt{(a - x)/(a + x)} \Rightarrow$ put $x = a\cos\theta$ or $a\cos2\theta$

5. $\dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}} \Rightarrow$ put $x = a\sin2\theta$

If $y = \tan^{-1}\frac{3x - x^3}{1 - 3x^2}$, then dy/dx equals-

(A) $3x$ (B) $\tan 3x$ (C) $\frac{3}{1+x^2}$ (D) $3\tan^{-1}x$

$\tan^{-1}\frac{3x-x^3}{1-3x^2}$ $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right) = \tan^{-1}\tan 3\theta = 3\theta = 3\tan^{-1}x$

$y = 3\tan^{-1}x.$

$\frac{dy}{dx} = 3 \times \frac{1}{1+x^2}$

If $y = \tan^{-1}\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, then $\dfrac{dy}{dx}$ equals-

(A) $-\dfrac{1}{2\sqrt{1-x^2}}$ (B) $-\dfrac{1}{\sqrt{1-x^4}}$ (C) $-\dfrac{x}{\sqrt{1-x^4}}$ (D) $-\dfrac{x}{2\sqrt{1-x^4}}$

$1+\cos 2\theta = 2\cos^2\theta$

$1-\cos 2\theta = 2\sin^2\theta$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right), \text{ where } x^2 = \cos\theta$$

$\theta = \cos^{-1}x^2$

$$\tan^{-1}\frac{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}} = \tan^{-1}\left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\theta/2}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right) = \tan^{-1}\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)$$

$$= \tan^{-1}[\tan(\pi/4+\theta/2)] = \boxed{\pi/4+\theta/2} = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$$\therefore \frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{1-x^4}}\cdot 2x = -\frac{x}{\sqrt{1-x^4}}$$

$\log mn = \log m + \log n$

T-6 LOGARITHMIC DIFFERENTIATION :

$y = e^x \times \sin x \times 2^x \times x^2$

$\log y = \log(e^x \times \sin x \times 2^x \times x^2)$

$= \log e^x + \log \sin x + \log 2^x + \log x^2$

To find the derivative of :

(i) a function which is the product or quotient of a number of functions

$$y = f_1(x)\, f_2(x)\, f_3(x) \ldots\ldots \quad \text{or} \quad y = \frac{f_1(x)\, f_2(x)\, f_3(x) \ldots}{g_1(x)\, g_2(x)\, g_3(x) \ldots}$$

(ii) a function of the form $[f(x)]^{g(x)}$ where f & g are both derivable, it will be found convenient to take the logarithm of the function first & then differentiate

OR

$y = (f(x))^{g(x)} = e^{g(x).\ln(f(x))}$ and then differentiate.

$(\text{var})^{\text{var}}$ $\boxed{(f(x))^{g(x)}}$

$f(x) = e^{\ln f(x)}$

$e^{\ln (f(x))^{g(x)}}$

$e^{g(x) \ln f(x)}$

Find $\frac{d}{dx}\left(x^{x^x}\right)$.

$y = x^{x^x}$

$\log y = \log x^{x^x}$

$\log y = x^x \log x$

$\frac{1}{y}\frac{dy}{dx} = x^x \times \frac{1}{x} + \log x\left(\frac{d}{dx} x^x\right)$

$\frac{dy}{dx} = y\left[\frac{x^x}{x} + \log x \times x^x(1+\log x)\right]$

$\frac{d}{dx} x^x$ $\quad (\sin x)^x \quad (x)^{\sin x}$

$y = x^x$

$\log y = \log x^x$

$\log y = x \log x$

$\frac{1}{y} \times \frac{dy}{dx} = x \times \frac{1}{x} + \log x * 1$

$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$

$\frac{dy}{dx} = y(1+\log x) = x^x(1+\log x)$

Find $\frac{d}{dx}\left(x^{x^x}\right)$.

Let $y = x^{x^x} \Rightarrow \log y = x^x \log x$

$$\frac{1}{y}\frac{dy}{dx} = x^x.\frac{1}{x} + \log x\,[x^x(1+\log x)]$$

$$\frac{dy}{dx} = x^{x^x}\,[x^{x-1} + \log x\,\{x^x(1+\log x)\}]$$

$y = f(x)$ explicit form

T-7 IMPLICIT FUNCTIONS :

If the variable x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible to express y as a function of x in the form $y = \phi(x)$, then such functions are said to be implicit functions.

DERIVATIVE OF IMPLICIT FUNCTION :

(i) In order to find dy/dx, in the case of implicit functions, we differentiate each term w.r.t. x regarding y as a functions of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit functions, both x & y are present.

A DIRECT FORMULA FOR IMPLICIT FUNCTIONS :

Let $f(x, y) = 0$. Take all the terms of left side and put left side equal to $f(x, y)$.

$$\text{Then } \frac{dy}{dx} = -\frac{\text{diff. of f w.r.t. x keeping y as constant}}{\text{diff. of f w.r.t. y keeping x as constant}}$$

If $x^2 + y^2 = 4x + 2y$ then dy/dx equals–

(A) $\frac{2-x}{y-1}$ (B) $\frac{x-2}{y-1}$ (C) $\frac{y-1}{x-2}$ (D) $\frac{x}{y}$

$$2x + 2y\frac{dy}{dx} = 4 + 2\frac{dy}{dx}$$

$$\frac{dy}{dx}(2y-2) = 4-2x \Rightarrow \frac{dy}{dx} = \frac{4-2x}{2y-2} = \frac{2-x}{y-1}$$

If $x^2 + y^2 = 4x + 2y$ then dy/dx equals–

(A) $\dfrac{2-x}{y-1}$ (B) $\dfrac{x-2}{y-1}$ (C) $\dfrac{y-1}{x-2}$ (D) $\dfrac{x}{y}$

M-2

$$\boxed{x^2+y^2-4x-2y}=0$$

$$g(x,y) = x^2+y^2-4x-2y$$

$$\frac{dy}{dx} = -\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}} = -\frac{2x-4}{2y-2} = -\frac{(x-2)}{y-1} = (2-x)/y-1$$

If $y^5 + xy^2 + x^3 = 4x + 3$, then find $\frac{dy}{dx}$ at (2, 1)

$$y^5 + xy^2 + x^3 - 4x - 3 = 0$$

$$\frac{dy}{dx} = -\frac{(y^2 + 3x^2 - 4)}{5y^4 + 2xy}$$

$x = 2,\ y = 1$

$$= (-1)$$

DERIVATIVE OF A FUNCTION W.R.T. ANOTHER FUNCTION

Let $y = f(x)$; $z = g(x)$ then $\frac{dy}{dz} = \frac{dy/dx}{dz/dx} = \frac{f'(x)}{g'(x)}$.

$$\boxed{\frac{dy}{dz} = \frac{dy/dx}{dz/dx}} = \frac{f'(x)}{g'(x)}.$$

If $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, then dy/dx equals-

(A) $\tan\theta$ (B) $\cot\theta$ (C) $\tan\frac{1}{2}\theta$ (D) $\cot\frac{1}{2}\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(\sin\theta)}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cos^2\theta/2} = \tan\frac{\theta}{2}$$

If $y = \sin^{-1}\dfrac{2x}{1+x^2}$, $z = \tan^{-1}x$, then the value of dy/dz is-

(A) $\dfrac{1}{1+x^2}$ (B) $\dfrac{2}{1+x^2}$ (C) 2 (D) None of these

$y = \sin^{-1}\dfrac{2x}{1+x^2}$ $x = \tan\theta$

$= \sin^{-1}\dfrac{2\tan\theta}{1+\tan^2\theta}$

$= \sin^{-1}\sin 2\theta$

$= 2\theta$

$\boxed{y = 2\tan^{-1}x}$

$z = \tan^{-1}x.$

$\dfrac{dy}{dz} = \dfrac{dy/dx}{dz/dx} = \dfrac{2 \times \frac{1}{\cancel{1+x^2}}}{\frac{1}{\cancel{1+x^2}}} = 2$

DERIVATIVES OF ORDER TWO & THREE

the second derivative of y w. r. t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Similarly, the 3rd order derivative of y w. r. t. x , if it exists, is defined by $\boxed{\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)}$ It is also denoted by $f'''(x)$ or y'''.

$y = \sin x$

$\frac{dy}{dx} = \cos x$

$\boxed{\frac{d}{dx}\left(\frac{dy}{dx}\right)} = \frac{d}{dx}\cos x$

$\frac{d^2y}{dx^2} = -\sin x$

f(x) is a function such that f '' (x) = – f(x) and f ' (x) = g(x) and h(x) is a function such that h(x) = $[f(x)]^2 + [g(x)]^2$ and h(5) = 11, then the value of h (10) is-

(A) 0 (B) 1 (C) 10 (D) None of these

$f''(x) = -f(x)$ $f'(x) = g(x) \Rightarrow g'(x) = f''(x)$

$h(x) = [f(x)^2] + g(x))^2$

$h'(x) = 2f(x)\cdot\boxed{f'(x)} + 2g(x)\cdot\boxed{g'(x)}$

$= 2f(x)\cdot g(x) + 2g(x)\, f''(x)$

$= 2f(x)\cdot g(x) + 2g(x)(-f(x))$

$= \cancel{2f(x)\cdot g(x)} - \cancel{2g(x)}\, f(x)$

$h'(x) = 0$ $\Rightarrow h(x) = 11$ $h(10) = 11$

$h(x) = K.$

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The value of the derivative of $|x-1| + |x-3|$ at $x = 2$ is-

(A) -2 (B) 0 (C) 2 (D) Not defined

$y = |x-1| + |x-3|$

$x = 2$

$= x-1 + -(x-3)$

$= x-1-\cancel{x}+3$

$\boxed{y = 2}$

$\frac{dy}{dx} = 0$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

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If $f(x) = \sqrt{1 - \sin 2x}$, then $f'(x)$ is equals to -

(1) $-(\cos x + \sin x)$, for $x \in (\pi/4, \pi/2)$

(2) $(\cos x + \sin x)$, for $x \in (0, \pi/4)$

(3) $-(\cos x + \sin x)$, for $x \in (0, \pi/4)$ ✓

(4) $\cos x - \sin x$, for $x \in (\pi/4, \pi/2)$

$$f(x) = \sqrt{1 - \sin 2x} = \sqrt{\sin^2 x + \cos^2 x - 2\sin x \cos x}$$

$$= \sqrt{(\sin x - \cos x)^2}$$

$$\boxed{\sqrt{x^2} = |x|}$$

$$f(x) = |\sin x - \cos x|$$

$0, \frac{\pi}{4}$

$\frac{\pi}{4}, \frac{\pi}{2}$

$$f(x) = \begin{cases} \cos x - \sin x & x \in (0, \frac{\pi}{4}) \\ \sin x - \cos x & x \in (\frac{\pi}{4}, \frac{\pi}{2}) \end{cases}$$

$$f'(x) = \begin{cases} -\sin x - \cos x & x \in (0, \frac{\pi}{4}) \\ \cos x + \sin x & x \in (\frac{\pi}{4}, \frac{\pi}{2}) \end{cases}$$