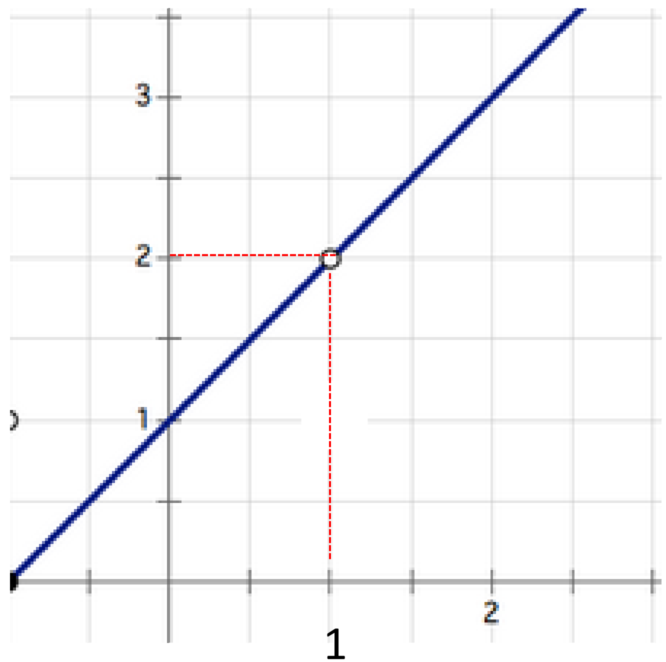
LIMIT: Basic Limit Definition

Let f(x) be defined in an open interval about **x=a**

BUT Not defined at **x=a**

Then, if for all x close to '**a**', f(x) will get an arbitrarily value close to a unique number l.

$$let\ f(x) = \frac{x^2-1}{x-1}$$



f(1) is undefined

$$f(x) = \frac{(x-1)(x+1)}{(x-1)} = x + 1$$

$$\lim_{x\to 1} f(x) = 2$$

LIMIT: LHL and RHL

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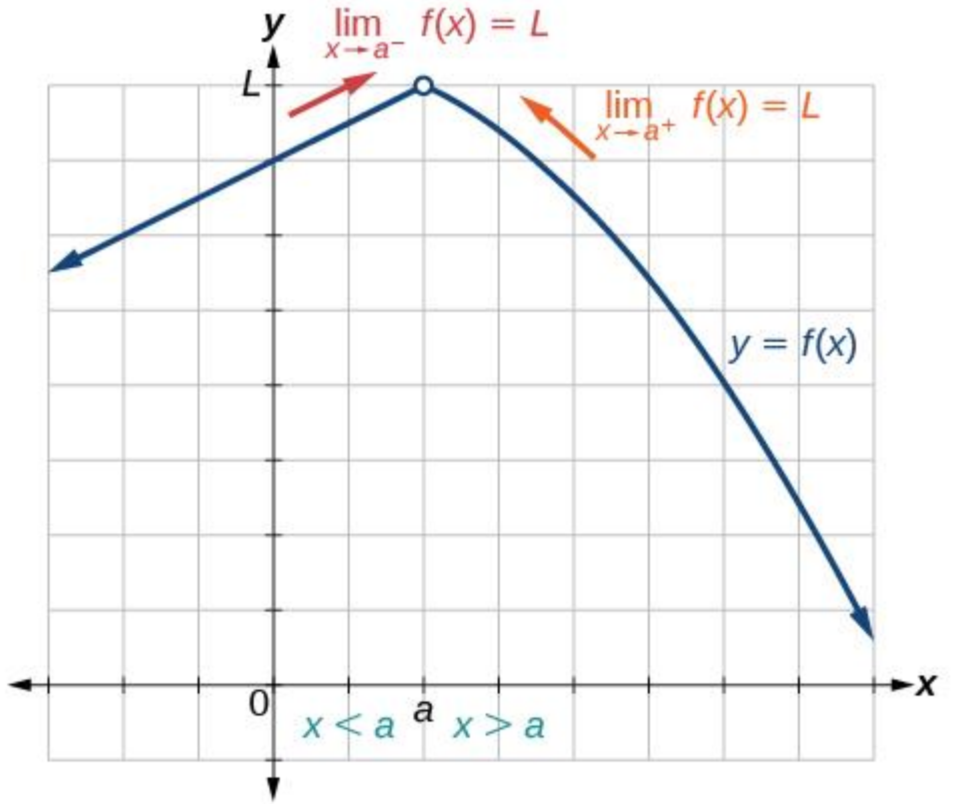

Left hand limit(L.H.L) = $\lim_{x \to a^-} f(x)$

Put x= a - h

L.H.L = $\lim_{h \to 0^+} f(a - h)$

Right hand limit (R.H.L) = $\lim_{x \to a^+} f(x)$

Put x= a + h

R.H.L= $\lim_{h \to 0^+} f(a + h)$

Limit exits if and only L.H.L = R.H.L

Therefore, you will have to check both the limits (LHL and RHL)

Theorems of Limit: Limit Application to Fraction

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Limit can be applied to both Numerator and Denominator

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

Denominator should be a **Non zero** limit

$$ex: \quad \lim_{x\to 0}\frac{\mathbf{1+sinx}}{\mathbf{2cos^2x}}$$

Note here → Cos 0 is not zero;

Therefore we could applied Limit like this.

$$=\frac{\lim_{x\to 0} 1+sinx}{\lim_{x\to 0} 2cos^2x}$$

$$=\frac{1}{2}$$

$$\lim_{x\to a}(f(x))^n=(\lim_{x\to a} f(x))^n$$

Note here, there is no condition on ‘**n**’

ex:

$$\lim_{x\to 0}\left(\frac{2sinx}{x}\right)^n=2^n$$

Cancellation of factor

Find out terms in a factor where denominator is becoming zero at Limit point;

So that Function remains defined at Limit ‘**x**’ value

Example:

$$f(x) = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)}$$

$$f(x) = \lim_{x \to 2} \frac{(x^2-4)(x+2)(x-5)}{(x^2-4)}$$

See here, the problem was denominator was zero at Limit ‘x’ Value

Method to Solve Limit by Standard Formulae

Standard formulae

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2. $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

3. $\lim_{x \to 0} \frac{tan x}{x} = 1$

4. $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

These are all

$\frac{0}{0}$ form

But their limit value is 1

Because in expansion of **sin x** & **tan x**, the first term is ‘x’

Method to Solve Limit by Standard Formulae

Standard formulae

1. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

2. $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a$

3. $\lim_{x\to 0} \frac{(1-x)^n - 1}{x} = -n$

4. $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

These are all

$\frac{0}{0}$ form

Here, in <u>expansion of the numerator</u>,

the first term will get divided by

denominator;

Then it will give a non-zero value

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$\frac{\infty}{\infty}$ form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Think like to divide numerator and denominator by largest term

when x →a , $f(x) \to \pm\infty$, $g(x) \to \pm\infty$

$$ex: \lim_{x \to \infty} \frac{x^2 + 1}{2x^2 - x + 2}$$

$$\Rightarrow \lim_{x \to \infty} \frac{1 + \frac{1}{x^2}}{2 - \frac{1}{x} + \frac{2}{x^2}}$$

See here $\boldsymbol{x^2}$ *is the largest term*

Therefore divide numerator and denominator by $\boldsymbol{x^2}$

Limit: Indeterminate Form $(0\times \infty)$ $form$

$(0\times \infty)$ $form$ → *convert* → $\frac{0}{0}$ or $\frac{\infty}{\infty}$ $form$

$$\lim_{x\to a} f(x).g(x)$$
$$f(x) \to 0$$
$$g(x) \to \infty$$

$$\lim_{x\to a} \frac{f(x)}{1/g(x)}$$

$\frac{0}{0}$ $from$

or

$$\lim_{x\to a} \frac{g(x)}{1/f(x)}$$

$\frac{\infty}{\infty}$ $from$

Remember:-

$$\lim_{x\to 0} x.\ln x = 0$$

$$\lim_{x\to 0} \frac{\ln x}{1/x}$$

L'Hospital Rule only applicable for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ $form$

Limit: Indeterminate Form 1^{∞}

1^{∞} *form*

$$L = \lim_{x \to a} (f(x))^{g(x)}$$

$as, x \to a$
$f(x) \to 1$
$g(x) \to \pm\infty$

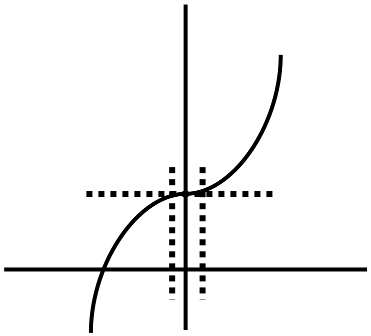
$$L = e^{\lim_{x \to a} (f(x)-1)g(x)}$$

Some particular case of 1^{∞} *form*	Final result
$\lim_{x \to 0} (1+x)^{\frac{1}{x}}$	e
$\lim_{x \to 0} (1+\lambda x)^{\frac{1}{x}}$	e^{λ}
$\lim_{x \to \infty} \left(1+\frac{1}{x}\right)^{x}$	e
$\lim_{x \to \infty} \left(1+\frac{\lambda}{x}\right)^{x}$	e^{λ}

Continuity

Continuity at an interior point

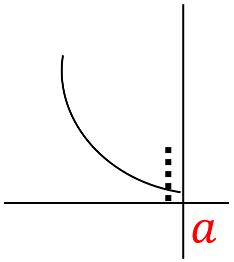
$$\lim_{x \to a} f(x) = f(a)$$



One-sided Continuity :-

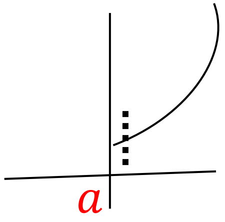
Left continuous at $x = a$

$$\lim_{x \to a^-} f(x) = f(a)$$



Right Continuous at $x = a$

$$\lim_{x \to a^+} f(x) = f(a)$$



Condition of Continuity ⟶ **L.H.L = f(a)= R.H.L**

Therefore, you will have to check both the limits (LHL and RHL) and interior point.

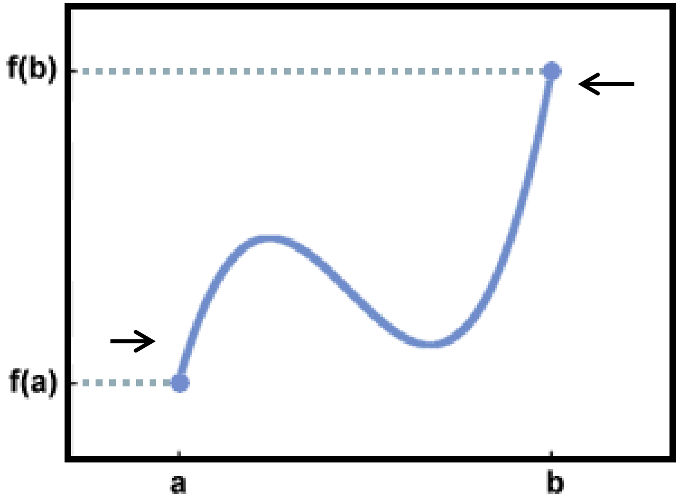
Continuity Over a Closed Interval

All elementry functions are continuous at each point in their Domain

Continuity over a closed interval $[a, b]$:-

1) Continuity over an open interval (a, b)

2) Right Hand continuous at $x = a$
$(\mathrm{R.H.L}) = f(a)$



3) Left Hand continuous at $x = b$
$(\mathrm{L.H.L}) = f(b)$

Types of Discontinuity:

1) **Removable Discontinuity** $\mathrm{L.H.L} = \mathrm{R.H.L} \neq f(a)$

2) **Irremovable Discontinuity** Limit of function does not exist

Condition for identifying type of discontinuity

. Let $f(x)=\begin{cases} -2\sin x & \text{if } x \le -\frac{\pi}{2} \\ A\sin x + B & \text{if } -\frac{\pi}{2} < x < \frac{\pi}{2}; \\ \cos x & \text{if } x \ge \frac{\pi}{2} \end{cases}$

For what values of A and B, the function $f(x)$ is continuous throughout the real line ?

(A) A = 1, B = 1 (B) A = − 1, B = 1

(C) A = − 1, B = − 1 (D) A = 1, B = − 1

Properties of Continuous Function

Let $\boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{g}(\boldsymbol{x})$ be continuous at $x = a$

The following function are continuous at $\boldsymbol{x = a}$

1) $y = kf(x)$

2) $y = (f(x))^{P}$

3) $y = f(x) \pm \boldsymbol{g}(\boldsymbol{x})$

4) $y = f(x).\,\boldsymbol{g}(\boldsymbol{x})$

5) $y = \dfrac{f(x)}{\boldsymbol{g}(\boldsymbol{x})}$, $\boldsymbol{g}(\boldsymbol{a}) \neq \boldsymbol{0}$

Properties of Continuous Function

If, at $x = a$

$f(x)$ continuous

$g(x)$ discontinuous

⇒ $f(x) \pm \boldsymbol{g(x)}$

discontinuous

$f(x)$ discontinuous

$g(x)$ discontinuous

⇒ $f(x) \pm \boldsymbol{g(x)}$ $\quad$ $f(x).\boldsymbol{g(x)}$, $\quad \dfrac{f(x)}{\boldsymbol{g(x)}}$

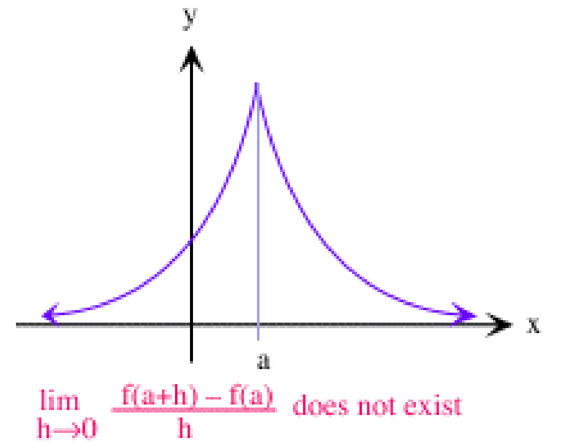
<u>may or may not be continuous</u>

Differentiability : L.H.D & R.H.D

Left hand derivative:

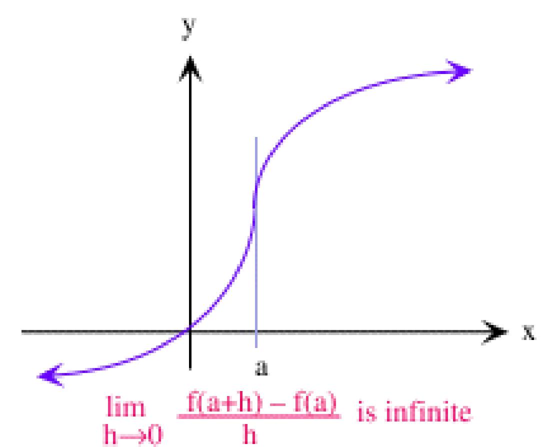
$$\mathbf{L.H.D} = \lim_{h \to 0} \frac{f(a) - f(a-h)}{a - (a-h)}$$

$$f'(a^-) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$



Right Hand Derivative:

$$f'(a^+) = \mathbf{R.H.D} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



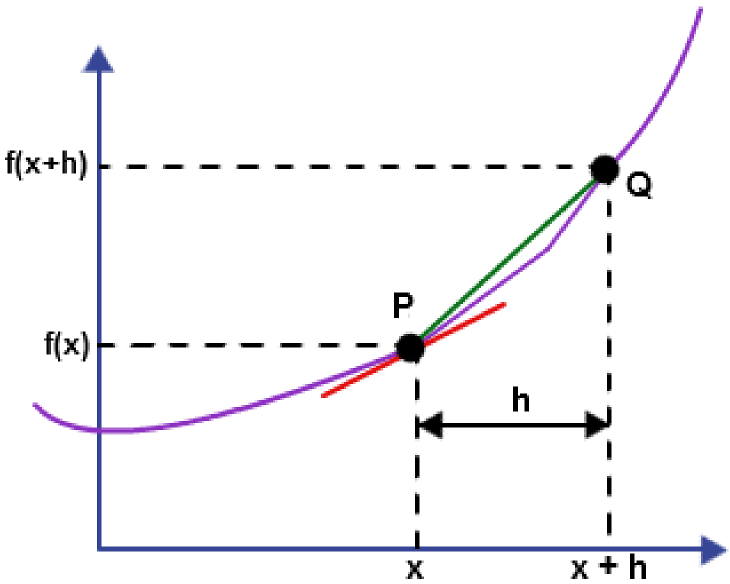
If $L.H.D = R.H.D$ then $y = f(x)$ is differentiable at $x = a$

Differentiability

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If you know function is differentiable then use this formula

Where,
h is small positive quantity



$$Ex: \ y = x^2 lnx$$

$$y' = \lim_{h \to 0} \frac{(x+h)^2 \ln(x+h) - x^2 lnx}{h}$$

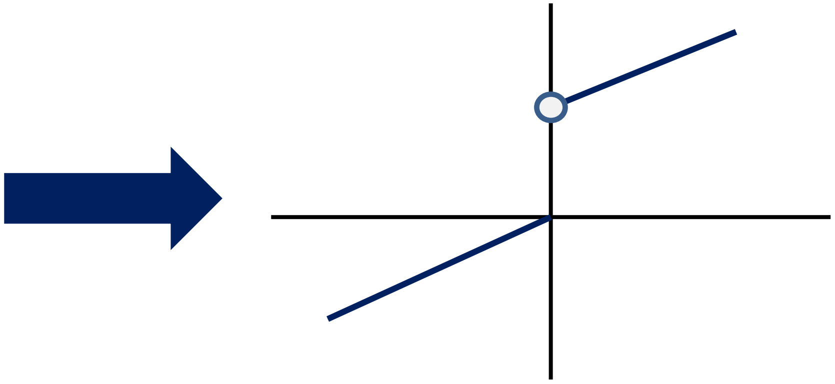
$$Ex: \ y = e^{\cos x}$$

$$y' = \lim_{h \to 0} \frac{e^{\cos(x+h)} - e^{\cos x}}{h}$$

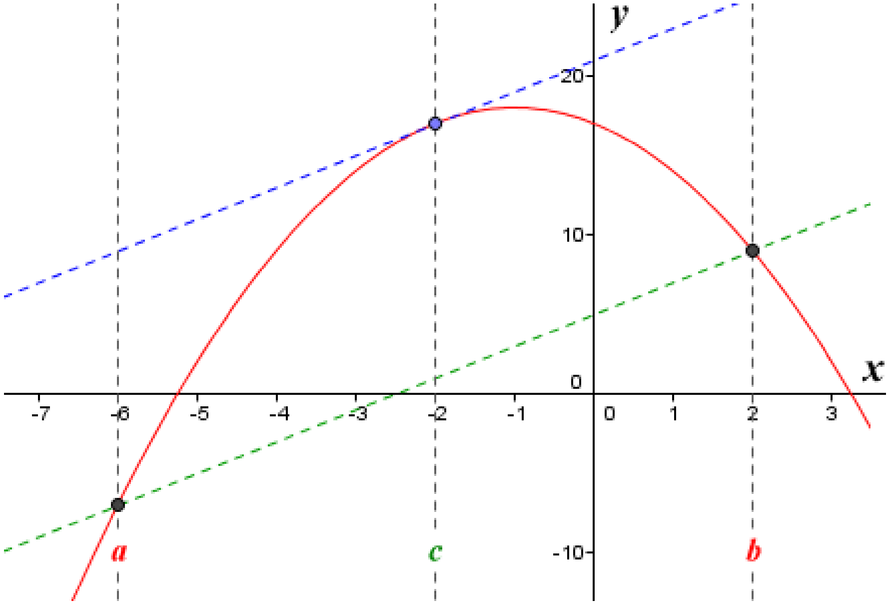
Relation Between Continuity and Differentiability Type I

1. If f(x) is **discontinuous** at $\boldsymbol{x = a}$ then f(x) is **non-differentiable** at $\boldsymbol{x = a}$

$$Ex: f(x) = x \quad , x \leq 0$$
$$= x + 1, x > 0$$



2. If f(x) is **differentiable** at $\boldsymbol{x = a}$, then f(x) is **continuous** at $\boldsymbol{x = a}$

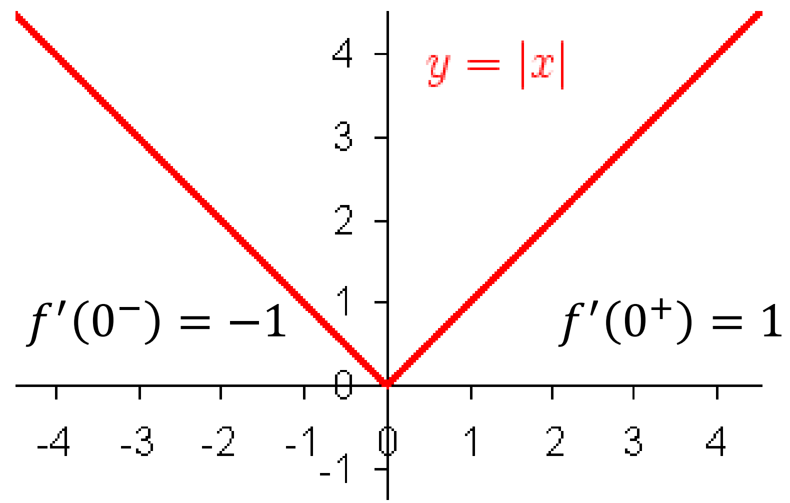


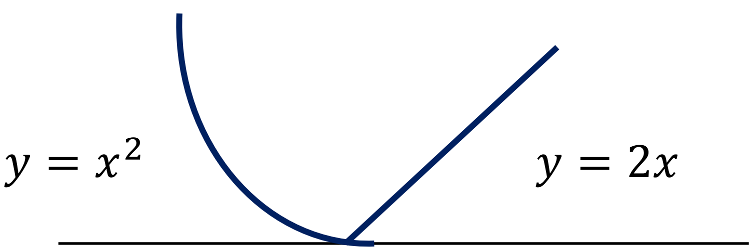
Relation Between Continuity and Differentiability Type II

If f(x) is **continuous** at $\boldsymbol{x = a}$ then f($\boldsymbol{x}$) may or may not be **differentiable** at $\boldsymbol{x = a}$

Corner

L. H. D and R. H. D are different but $f(x)$ is **continuous**





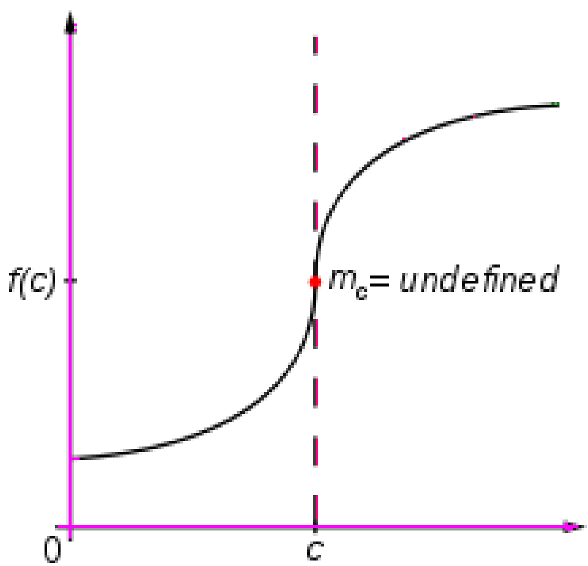
L. H. D = 0

R. H. D = 2

Cases of Non-Differentiability: Vertical Tangent

Vertical tangent

A **tangent** line that is **vertical**, meaning it has infinite slope and function whose graph has a **vertical tangent** is not differentiable at the point of tangency.



Vertical tangent on the function f(x) at x=c

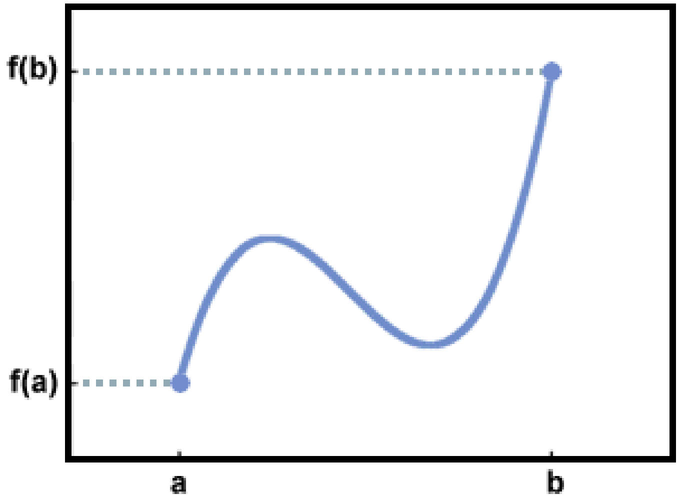
In this case $|f'(x)| \to \infty$ at x = c

Differentiability on a Closed Interval $[a, b]$

Check following condition

1. Differentiable on open interval (a, b)
2. Differentiable at $x = b$ (L. H. D)
3. Differentiable at $x = a$ (R. H. D)

Check all the conditions for differentiability over a closed interval



Differentiability : Standard Result of Differentiability

Standard Result of Differentiability

Functions $f(x)$	Interval in which $f(x)$ is $Differentiable$
Polynomial	$(-\infty, \infty)$
Exponential$(a^x, a > 0)$	$(-\infty, \infty)$
Constant	$(-\infty, \infty)$
Logarithmic	Each point in its domain
Trigonometric	Each point in its domain
Inverse Trigonometric Function	Each point in its domain