CONCEPT OF LIMITS :

Suppose f(x) is a real-valued function and c is a real number. The expression $\lim_{x\to c} f(x) = L$ means that f(x) can be as close to L as desired by making x sufficiently close to c. In such a case, we say that limit of f, as x approaches c, is L. Consider $\boxed{f(x) = x - 1}$ as x approaches 2.

$\lim_{x\to a} f(x)$ As x approaches 'a', if $f(x)$ approaches 'L' then we say limit of $f(x)$ is L

f(1.9)	f(1.99)	f(1.999)	f(2)	f(2.001)	f(2.01)	f(2.1)
0.9	0.99	0.999	$\rightarrow 1 \leftarrow$	1.001	1.01	1.1

As x aproaches 2, f(x) approaches 1 and hence we have $\boxed{\lim_{x\to 2} f(x) = 1}$

$f(x) = x-1$

$\lim_{x\to 2} f(x)$ $\qquad \ne \quad x \ne 2$

REMEMBER
$\text{Limit}_{x\to a} \Rightarrow x \neq a$

Left-and Right-Hand Limits :

Right hand limit of a function is that value of $f(x)$ which function tends as x moves from right to number 'a' that is RHL $= \lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a+h)$ (where $h > 0$)

Left hand limit of a function is that value of $f(x)$ which function tends x moves from left to number 'a'. that is LHL $= \lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h)$ (where $h > 0$)

$\lim\limits_{x \to a^-} f(x) = \lim\limits_{h \to 0} f(a-h)$

$x = a-h$

$\lim\limits_{x \to a^+} f(x) = \lim\limits_{h \to 0} f(a+h)$

$x = a+h$

Existence of Limit :

If follows from the discussions made in the previous two sections that $\lim_{x \to a} f(x)$ exists if $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exists and both are equal.

Thus, $\boxed{\lim_{x \to a} f(x) \text{ exists} \Leftrightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).}$

Determine the following limits.

(a) $\lim\limits_{x\to -1^+} f(x)$ $=2$

(b) $\lim\limits_{x\to -1^-} f(x)$ $=1$

(c) $\lim\limits_{x\to -1} f(x)$ $=$ D.N.E

(d) $\lim\limits_{x\to 1^+} f(x)$ $=2$

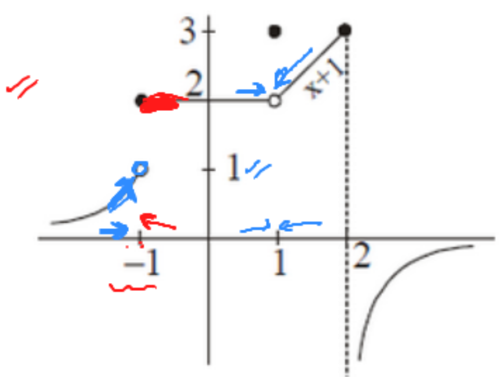
(e) $\lim\limits_{x\to 1^-} f(x)$ $=2$

(f) $\lim\limits_{x\to 1} f(x)$ $=2$

(g) $\lim\limits_{x\to 2^+} f(x)$

(h) $\lim\limits_{x\to 2^-} f(x)$

(i) $\lim\limits_{x\to 2} f(x)$



If $f(x) = \begin{cases} x^2+1 & x \geq 1 \\ 3x-1 & x < 1 \end{cases}$ then the value of $\lim_{x \to 1} f(x)$ is -

(A) 1 (B) 2 (C) 3 (D) Does not exist

$\lim_{x \to 1} f(x) = ?$

$L.H.L = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x-1) = 3 \times 1 - 1 = 2$

$R.H.L = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2+1) = 1^2+1 = 2$

The value of $\lim_{x \to 1} [x]$ is-

(A) 1 (B) 2 (C) 4 (D) Does not exist

$\lim_{x \to 1} [x]$

$L.H.L = \lim_{x \to 1^-} [x] = 0$

$R.H.L = \lim_{x \to 1^+} [x] = 1$

$[0.99] = 0$

$[1.001] = 1$

THE ALGEBRA OF LIMITS :

Let $\lim\limits_{x \to a} f(x) = l$ and $\lim\limits_{x \to a} g(x) = m$. If l and m exist, then

1. $\lim\limits_{x \to a} (f \pm g)(x) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x) = l \pm m$

2. $\lim\limits_{x \to a} (fg)(x) = \lim\limits_{x \to a} f(x) \lim\limits_{x \to a} g(x) = l\,m$

3. $\lim\limits_{x \to a} \left(\dfrac{f}{g}\right)(x) = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

4. $\lim\limits_{x \to a} k\,f(x) = k.\lim\limits_{x \to a} f(x)$, where k is a constant

INDETERMINATE FORMS :

Indeterminant forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty - \infty$, $0 \times \infty$, 1^{∞}, 0^0 and ∞^0

Approaching values

$\lim_{x \to a} \frac{f(x)}{g(x)}$

$\lim_{x \to a} f(x) = 0$

$\lim_{x \to a} g(x) = 0$

Note : (i) Here 0,1 are not exact, infact both are aproaching to their corresponding values.

(ii) We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number It does not obey the laws of elementary algebra,

(a) $\infty + \infty \to \infty$ (b) $\infty \times \infty \to \infty$ (c) $\infty^{\infty} \to \infty$ (d) $0^{\infty} \to 0$

EVALUATION OF ALGEBRAIC LIMITS :

Direct Substitution Method :

Consider the following limits : $\lim\limits_{x \to a} f(x)$ and f(x) is continuous at x = a

then we say that $\lim\limits_{x \to a} f(x) = f(a)$

Evaluate : (i) $\lim\limits_{x \to 1} 3x^2 + 4x + 5$

$= 3 + 4 + 5$

$= 12$

(ii) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x + 3}$

where discontinuous

$\dfrac{2^2 - 4}{2+3} = 0$

When x $\to \infty$ $\frac{1}{x} \to 0$

In this case expression should be expressed as a function of 1/x and then after removing indeterminant form, (If it is there) replace 1/x by 0.

Find the value of $\lim_{x\to\infty} \frac{2x^3+4x^2+5}{9x^3+4x^2+7}$

$$\lim_{x\to\infty} \frac{2x^3+4x^2+5}{9x^3+4x^2+7} = \lim_{x\to\infty} \frac{2+\frac{4}{x}+\frac{5}{x^3}}{9+\frac{4}{x}+\frac{7}{x^3}}$$

$$= \frac{2}{9}$$

Divide Nr & Dr by x^3

$$\lim_{x\to\infty} \frac{2+\frac{4}{x}+\frac{5}{x^3}}{9+\frac{4}{x}+\frac{7}{x^3}} = \frac{2}{9}$$

When x $\to \infty$

In this case expression should be expressed as a function of 1/x and then after removing indeterminant form, (If it is there) replace 1/x by 0.

Find $\lim\limits_{x\to\infty} \dfrac{\sqrt{x+2}}{\sqrt{x+\sqrt{x}}+\sqrt{x+5}}$

$$\lim_{x\to\infty} \frac{\sqrt{x+2}}{\sqrt{x+\sqrt{x}}+\sqrt{x+5}} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{\cancel{2}}{x}}}{\sqrt{1+\sqrt{\frac{1}{x}}}+\sqrt{1+\frac{\cancel{5}}{x}}}$$

$$= \frac{\sqrt{1+0}}{\sqrt{1+\sqrt{0}}+\sqrt{1+0}} = \frac{1}{2}$$

Multiply & Divide by $\sqrt{x}$

$x \to \infty$

$\frac{1}{2}$

Factorisation :

If f(x) is of the form $\frac{h(x)}{g(x)}$ and of indeterminate form $\frac{0}{0}$ then this form is removed by factorising g(x) and h(x) and cancel the common factors, then put the value of x.

The value of $\lim_{x \to 2} \left(\frac{2x^2 - 7x + 6}{5x^2 - 11x + 2} \right)$ is -

$\frac{2 \times 2^2 - 7 \times 2 + 6}{5 \times 2^2 - 11 \times 2 + 2}$ $\quad \frac{0}{0}$ $\quad (x-2)$ is a factor, Nr & Dr

(A) 1/2 (B) 1/4 (C) 1/9 (D) Does not exist

$$\lim_{x \to 2} \left(\frac{2x^2 - 7x + 6}{5x^2 - 11x + 2} \right) \left(\frac{0}{0} \text{ form} \right) = \lim_{x \to 2} \frac{(x-2)(2x-3)}{(x-2)(5x-1)} = \lim_{x \to 2} \left(\frac{2x-3}{5x-1} \right) = \frac{4-3}{10-1} = 1/9$$ **Ans.[C]**

Factorisation :

If f(x) is of the form $\frac{h(x)}{g(x)}$ and of indeterminate form $\frac{0}{0}$ then this form is removed by factorising g(x) and h(x) and cancel the common factors, then put the value of x.

Find the value of $\lim_{x\to 0}\frac{\cos^2 x+\cos x-2}{\sin^2 x}$. $\to 1-\cos^2 x$

$$= \lim_{x\to 0}\frac{\cos^2 x+2\cos x-\cos x-2}{1-\cos^2 x} = \lim_{x\to 0}\frac{(\cos x+2)(\cos x-1)}{(1+\cos x)(1-\cos x)}$$

$$= \lim_{x\to 0}\frac{-(\cos x+2)}{1+\cos x} = -\frac{3}{2}$$

$\cos x = y \qquad x\to 0,\ y\to 1$

$$\lim_{y\to 1}\frac{y^2+y-2}{1-y^2} = \frac{(y+2)\cancel{(y-1)}}{-\cancel{(y-1)}(y+1)} = \frac{y+2}{-(y+1)} = \frac{3}{-2}$$

Rationalisation :

In this method we rationalise the factor containing the square root or cube root and simplify and then put the value of x.

$$\frac{1}{\sqrt{3}-\sqrt{2}} \times \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}$$

The value of $\lim\limits_{x \to 0} \dfrac{\sqrt{(1+x^2)} - \sqrt{1-x^2}}{x^2}$ is- $\frac{0}{0}$

(A) 1 (B) 2 (C) 3 (D) Does not exist

$$\lim_{x \to 0} \left(\frac{\sqrt{1+x^2} - \sqrt{1-x^2}}{x^2}\right) \cdot \frac{\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)}{\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)}$$

$$= \lim_{x \to 0} \frac{1+x^2 - 1 + x^2}{x^2\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)}$$

$$= \boxed{\frac{2\cancel{x^2}}{\cancel{x^2}\left(\sqrt{1+x^2} + \sqrt{1-x^2}\right)}} = \frac{2}{\left(\sqrt{1} + \sqrt{1}\right)} = \frac{2}{2} = 1$$

ALGEBRAIC FUNCTION

1. $\boxed{\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}}$, if a > 0, n ∈ R and if a < 0, n ∈ Z.

$\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$

2. $\lim_{x\to a}\frac{x^m-a^m}{x^n-a^n}=\frac{m}{n}a^{m-n}$

If $\lim_{x\to 5}\frac{x^k-5^k}{x-5}=500$, then the positive integral value of k is-

(A) 3 (B) 4 (C) 5 (D) 6

Here the given Limit
$= k\ 5^{k-1} = 500 \Rightarrow k.5^{k-1} = 4.5^3 \Rightarrow k = 4$

$k \cdot 5^{k-1} = 500$

$\frac{k \cdot 5^k}{5} = 500$

$k \cdot 5^k = 2500$

TRIGONOMETRIC FUNCTION

1. $\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x}{\sin x} = \lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{x}{\tan x} = 1$ $\quad \to 0/0$

2. $\lim_{x\to 0}\cos x = \lim_{x\to 0}\sec x = 1$

3. $\lim_{x\to 0}\frac{\sin^{-1}x}{x} = \frac{\tan^{-1}x}{x} = 1$

4. $\boxed{\lim_{x\to\infty}\frac{\sin x}{x} = 0}$ $\quad \lim_{x\to\infty}\frac{\sin x}{x}$ $\quad \frac{\sin\infty}{\infty} \to \frac{[-1,1]}{\infty} \to 0$

5. $\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 1$ $\quad x\to\infty \quad \frac{1}{x}\to 0$

$\frac{1}{x} = y$

$\lim_{y\to 0}\frac{\sin y}{y} = 1$

6. $\lim_{x\to 0}\left[\frac{\sin x}{x}\right] = 0$, where [.] represent step function

7. $\lim_{x\to 0}\left[\frac{x}{\sin x}\right] = 1$

8. $\lim_{x\to 0}[\cos x] = 0$

9. $\lim_{x\to 0}\left[\frac{\tan x}{x}\right] = 1$

At $x \to 0$

$\boxed{\sin x < x < \tan x}$

$\sin x < x < \tan x \Rightarrow \frac{\tan x}{x} > 1$

$\boxed{\frac{\sin x}{x} < 1} = 0.9999$

$\frac{x}{\sin x} > 1$

$\left[\frac{x}{\sin x}\right] = [1.00001] = 1$

10. $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$

$\frac{2\sin^2 x/2}{4 \quad x^2/4} = \frac{1}{2}\left(\frac{\sin\frac{x}{2}}{x/2}\right)^2$

$= \frac{1}{2}$

11. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

12. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

13. $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

Find $\lim\limits_{x \to \frac{\pi}{2}} \frac{2x-\pi}{\cos x}$ $= \frac{0}{0}$ $x = \frac{\pi}{2} + h.$ =

$\lim\limits_{x \to \frac{\pi}{2}} \frac{2x-\pi}{\cos x}$, Put $x = \frac{\pi}{2} + h$ when $x \to \frac{\pi}{2}$, $h \to 0$

$$\text{Limit} = \lim_{h \to 0} \frac{2\left(\frac{\pi}{2} + h\right) - \pi}{\cos\left(\frac{\pi}{2} + h\right)} = \lim_{h \to 0} \frac{-2h}{\sin h} = -2$$

EXPONENTIAL & LOGARITHMIC FUNCTION

1. $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$

2. $\lim_{x \to 0} \frac{\log_e(1+x)}{x} = 1$ and $\lim_{x \to 0} \frac{\log_e(1-x)}{x} = -1$

3. $\lim_{n \to \infty} x^n = \begin{cases} 0, & 0 < x < 1 \\ 1, & x = 1 \\ \infty, & x > 1 \end{cases}$

$(0.5)^\infty \to 0$

$1^\infty \to 1$

$(1.5)^\infty \to \infty$

4. $a^x = 1 + x \log a + \frac{x^2}{2!}(\log a)^2 + \frac{x^3}{3!}(\log a)^3 + \ldots$

5. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$\lim_{n \to \infty} x^n$ $\swarrow \infty$

Find the following limits

(i) $\lim_{x \to 0} \frac{3^x - 1}{\tan x}$ $\frac{x}{x}$ (ii) $\lim_{x \to 3} \frac{\sin(e^{x-3} - 1)}{\log(x-2)}$

(i) $\lim_{x \to 0} \frac{3^x - 1}{\tan x} = \lim_{x \to 0} \boxed{\frac{3^x - 1}{x}} \times \boxed{\frac{x}{\tan x}} = \log 3 \times 1 = \log 3$

$\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$

$\lim_{x \to 0} \frac{x}{\tan x} = 1$

$\boxed{\text{Evaluation of } 1^\infty :}$ → indeterminate forms.

1. $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

2. $\lim_{x\to\infty}(1+x)^{1/x} = e$

$\lim_{x\to a} f(x) = 1$

$\lim_{x\to a} g(x) = \infty$

$\lim_{x\to a} [f(x)]^{g(x)} \to 1^\infty$

$$\boxed{= e^{\lim_{x\to a}(f(x)-1)\,g(x)}}$$

$$\boxed{(f(x))^{g(x)} = e^{\log (f(x))^{g(x)}}}$$

$$= e^{g(x)\log f(x)}$$

$x = e^{\log x}$

General Method to solve :

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then $\lim_{x\to a}[1+f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$

(ii) $\boxed{\text{If } \lim_{x\to a} f(x) = 1, \lim_{x\to a} g(x) = \infty \text{ then } \lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a}[f(x)-1]\times g(x)}}$

Find the value of following limits

(i) $\lim_{x \to 0} (1+3x)^{\frac{1}{5x}}$ (ii) $\lim_{x \to 0} \left(\frac{1+3x^2}{1+5x^2} \right)^{\frac{1}{x^2}}$ (iii) $\lim_{x \to 0} \left(\frac{1^x + 2^x + 3^x + \ldots n^x}{n} \right)^{\frac{1}{x}}$

$f(x)$

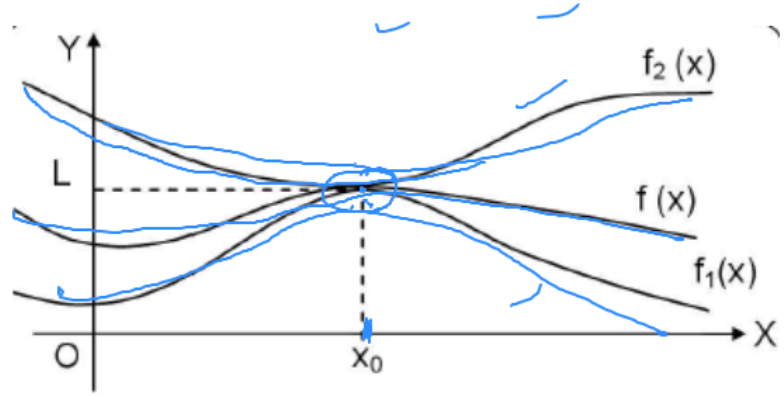
1^{∞}

$e^{\lim_{x \to 0} (1+3x-1) \times \frac{1}{5x}} = e$

$e^{\lim_{x \to 0} 3\cancel{x} \times \frac{1}{5\cancel{x}}} = e^{3/5}$

Sandwich theorem or squezze play theorem :

If there is a function h(x) such that $f(x) \le h(x) \le g(x)\ \forall\ x$ and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$ Then $\lim\limits_{x\to a} h(x) = \lim\limits_{x\to a} g(x)$ or $\lim\limits_{x\to a} f(x)$



$f(x)$

Evaluate $\lim\limits_{n\to\infty} \boxed{\dfrac{1}{1+n^2} + \dfrac{2}{2+n^2} + \ldots\ldots + \dfrac{n}{n+n^2}}$

$$\frac{1}{1+n^2} + \frac{2}{1+n^2} + \frac{3}{1+n^2} --- \frac{n}{1+n^2} \ge f(x) \ge \frac{1}{n+n^2} + \frac{2}{n+n^2} + \frac{3}{n+n^2} --- \frac{n}{n+n^2}$$

$$\frac{1+2+3 --- n}{1+n^2} \ge f(x) \ge \frac{1+2+3 --- n}{n+n^2}$$

$$\lim_{n\to\infty} \frac{n(n+1)}{2(1+n^2)} \ge f(x) \ge \frac{n(n+1)}{2(n+n^2)} \quad 1/2$$

$$\lim_{n\to\infty} f(x) = 1/2$$

1/2

L′ Hospital's Rule :

If f(x) and g(x) are function of x such that f(a) = 0 and g(a) = 0, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Here f '(x) and g'(x) both are differentiation of f(x) and g(x) w.r.t. x.

$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Forms $0 \times \infty$ and $\infty - \infty$, reduced either to $\boxed{\text{form } \frac{0}{0} \text{ or } \frac{\infty}{\infty}}$ for using L. Hospital's rule

0^0, 1^∞, ∞^0 such types of form can be reduced to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking log of the given limit.