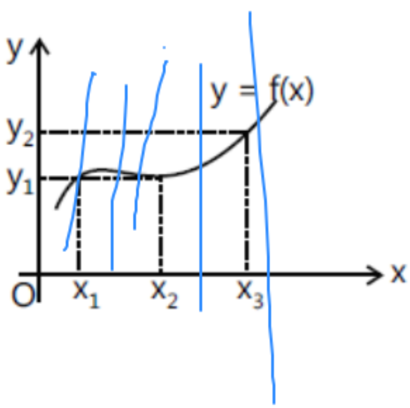
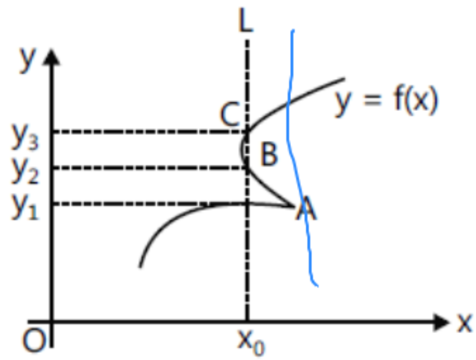
Functions

RELATION VS FUNCTION





Note :

(i) If a vertical line cuts a given graph at more than one point then it can not be the graph of a function.

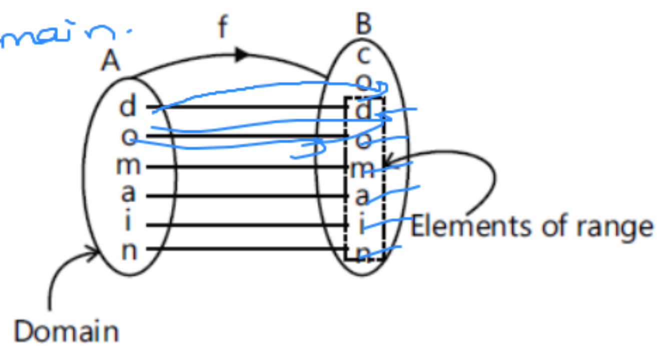
(ii) Every function is a relation but every relation is not necessarily a function.

Functions

DOMAIN, CO–DOMAIN & RANGE OF A FUNCTION :

Let $f: A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co-domain of f.
The set of all f images of elements of A is known as the range of f.

$f : A \rightarrow B$ → Codomain.



It should be noted that range is a subset of co–domain

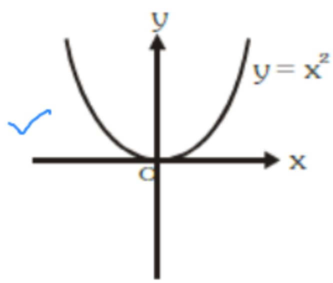
the domain of the function is the set of those real numbers, where function is defined.

For a continuous function, the interval from minimum to maximum value of a function gives the range.

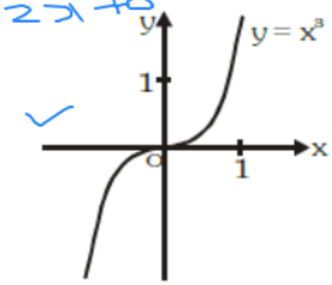
IMPORTANT TYPES OF FUNCTIONS

POLYNOMIAL FUNCTION :

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n .



$g(x) = 3x^3 + 5x^2 + 2x + 8$



	deg
Linear	1
cubic	3
	5

There are two polynomial functions, satisfying the relation ; f(x) f(1/x) = f(x) + f(1/x)

They are :

(i) $f(x) = x^n + 1$ and (ii) $f(x) = 1 - x^n$, where n is positive integer.

Domain of a polynomial function is R

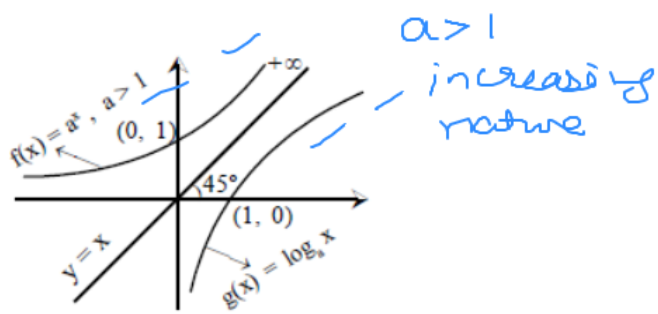
Range for odd degree polynomial is R whereas for even degree polynomial range is a subset of R.

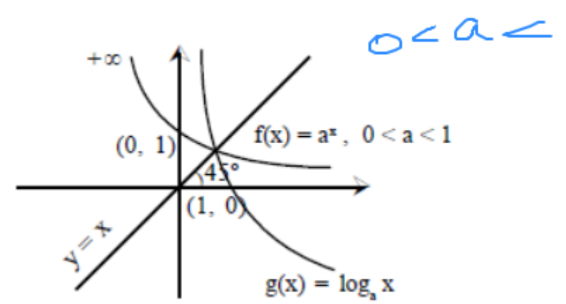
Exponential/Logarithmic Function

inverse

A function $f(x) = a^x = e^{x \ln a}$ $(a > 0,\ a \neq 1, x \in R)$ is called an exponential function. The inverse of the exponential function is called the logarithmic function . i.e. $g(x) = \log_a x$.

Note that $f(x)$ & $g(x)$ are inverse of each other & their graphs are as shown .





$f(x) = e^x$ domain is R and range is R^+.

Signum Function :

A function y= f (x) = Sgn (x) is defined as follows :

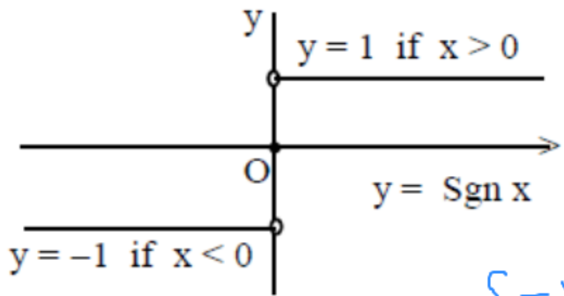
$$y = f(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

It is also written as $\mathrm{Sgn}\, x = |x| / x$;

$x \neq 0$; $f(0) = 0$

$+ve \to \mathrm{Sgn}(x) \to 1$

$-ve \to \mathrm{Sgn}(x) \to -1$



$\{-1, 0, 1\}$

$0 \to \mathrm{Sgn}(x) \to 0$

GREATEST INTEGER OR STEP UP FUNCTION :

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x. Note that for :

$-1 \le x < 0$	;	$[x] = -1$	$0 \le x < 1$	;	$[x] = 0$
$1 \le x < 2$	;	$[x] = 1$	$2 \le x < 3$	;	$[x] = 2$

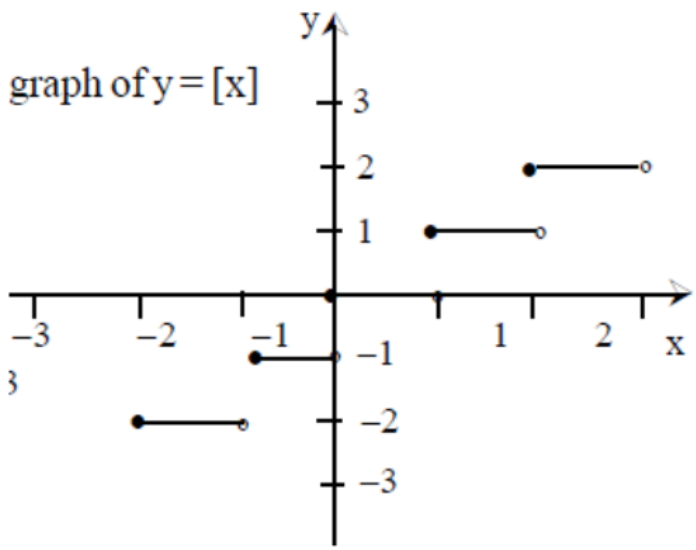
and so on .

$[3.4] = 3$

$[-3.4] = -4$

$[3] = 3$

$[-3] = -3$



Properties of greatest integer function :

(a) $[x] \le x < [x] + 1$ and
$x - 1 < [x] \le x$, $0 \le x - [x] < 1$

(b) $[x + m] = [x] + m$ if m is an integer .

(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

(d) $[x] + [-x] = 0$ if x is an integer
$= -1$ otherwise .

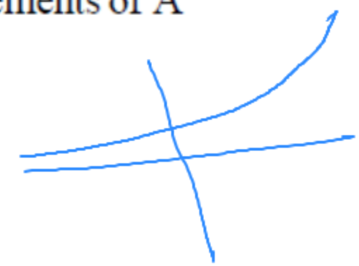
CLASSIFICATION OF FUNCTIONS :

One – One Function (Injective mapping) :

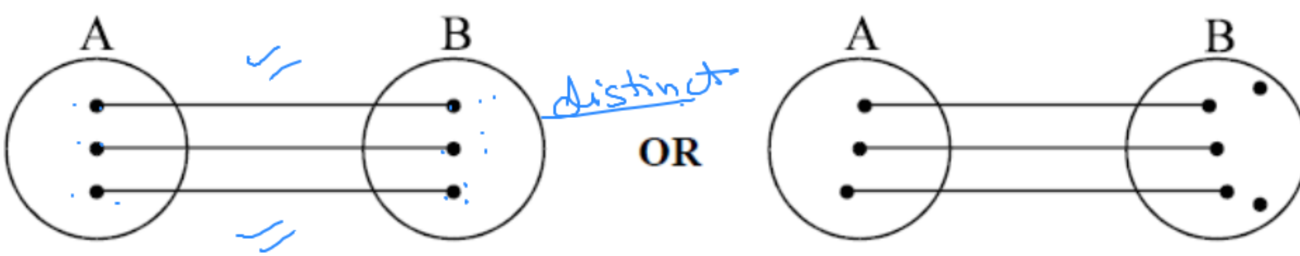
/ Many-One.

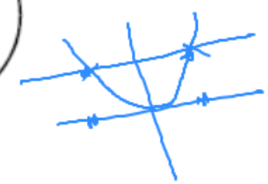
A function f: $A \rightarrow B$ is said to be a one–one function or injective mapping if different elements of A have different f images in B .

e^x



Diagramatically an injective mapping can be shown as





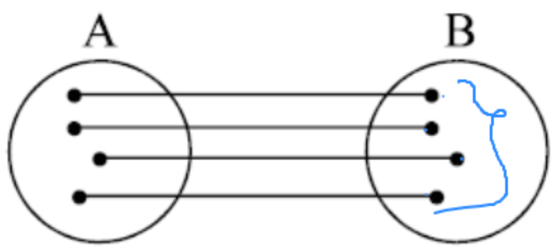
(i) Any function which is entirely increasing or decreasing in whole domain, then f(x) is one–one .

(ii) If any line parallel to x–axis cuts the graph of the function atmost at one point, then the function is one–one .

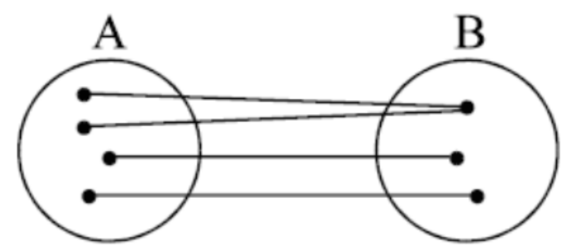
Onto function (Surjective mapping) :

If the function $f: A \rightarrow B$ is such that each element in B (co–domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B .

Diagramatically surjective mapping can be shown as



OR



Note that : if range = co–domain, then f(x) is onto.

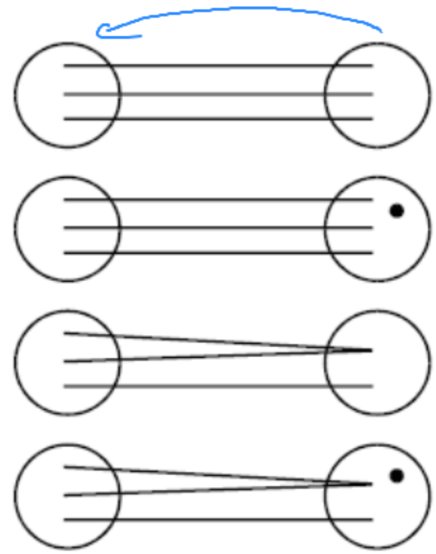
Thus a function can be one of these four types :

(a) one–one onto (injective & surjective) (bijective)

(b) one–one into (injective but not surjective)

(c) many–one onto (surjective but not injective)

(d) many–one into (neither surjective nor injective)



(i) If f is both injective & surjective, then it is called a **Bijective** mapping.
The bijective functions are also named as invertible functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n ! are one one.

COMPOSITE FUNCTIONS

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function gof : $A \to C$ defined by $(gof)(x) = g(f(x))\ \forall\ x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$.

$\boxed{f(x) = x^2}$

$f(g(x)) = (g(x))^2 = (\sin x)^2$

$= \sin^2 x$

$\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$

$g \circ f(x)$

$f(x) = \sin x$

$g(x) = \log x$

$h(x) = x^2$

$f(x) = x^2$, $g(x) = \sin x$

$f(g(x)) =$

Properties Of Composite Functions :

(i) The composite of functions is not commutative i.e. $gof \neq fog$.

$f \circ g \circ h$

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that fo(goh) & (fog)oh are defined, then fo(goh) = (fog)oh.

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

$g(f(x)) = \sin(f(x)) = \sin x^2$

COMPOSITE FUNCTIONS

Let $f: A \to B$ & $g: B \to C$ be two functions. Then the function $gof: A \to C$ defined by $(gof)(x) = g(f(x))\ \forall\ x \in A$ is called the composite of the two functions f & g.

Diagramatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$.

Find the domain

$g(f(x)) = \sqrt{2 - f(x)}$

$= \sqrt{2 - \sqrt{x}}$

If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2 - x}$, the composition $f \circ g(x)$ is

(a) $\sqrt{(2 - 2x)}$ (b) $(2 - x)^{\frac{1}{4}}$ (c) $x^{\frac{1}{4}}$ (d) $\sqrt{2 - \sqrt{x}}$

$f(g(x)) = \sqrt{g(x)} = \sqrt{\sqrt{2 - x}} = (2 - x)^{1/4}$

$2 - x \geq 0$

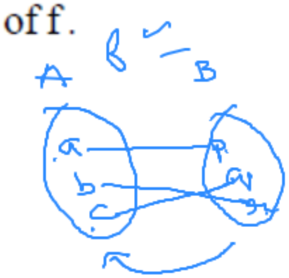
$x \leq 2$

$y = x^2$

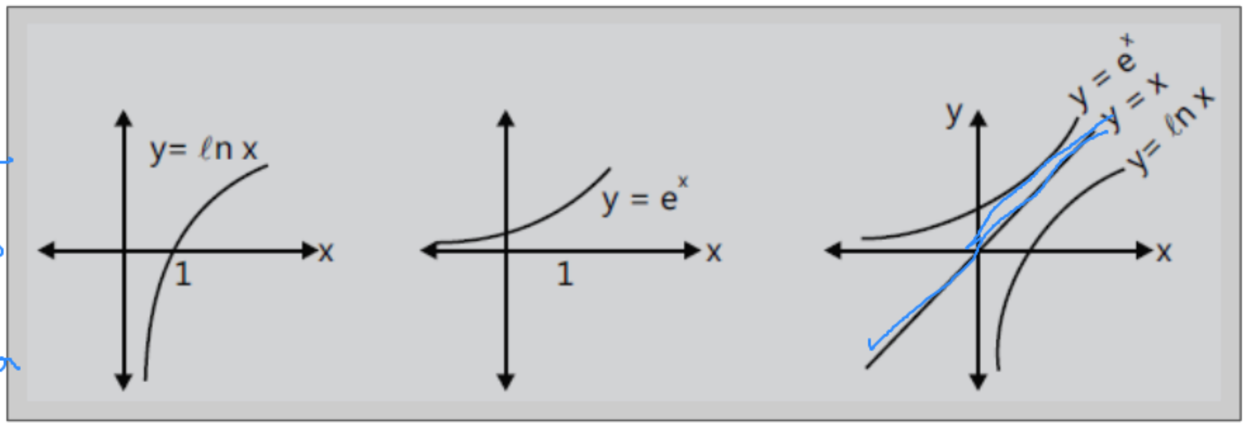
INVERSE OF A FUNCTION : $(5, 10) \in f$ $(10, 5) \in f^{-1}$

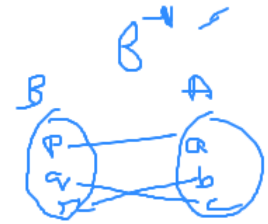
Let $f : A \to B$ be a one–one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A\ \&\ y \in B$. Then g is said to be inverse of f. bijective

$f(5) = 10$ $f^{-1}(10) = 5$



$*x$
$f \circ g = g \circ f = x$
when f & g
are inverse
of each other





Note that the graphs of f & g are the mirror images of each other in the line $\boxed{y = x}$.

If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $f \circ g = I_B$ and $g \circ f = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.

$f(f^{-1}(x)) = x$

$f^{-1}(f(x)) = x$

INVERSE OF A FUNCTION :

Let f: $A \to B$ be a one–one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A\ \&\ y \in B$. Then g is said to be inverse of f.

→ The inverse of a bijection is also a bijection.

If f & g are two bijections $f : A \to B,\ g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} o\, g^{-1}$.

$g(x) = e^x \quad (gof)^{-1} = f^{-1} \circ g^{-1}$

$g(f(x)) : A \to C$

$$\boxed{(g \circ f)^{-1} = f^{-1} \circ g^{-1}}$$

If $f(x) = \dfrac{6x-3}{2x+4}$, then $f^{-1}(x)$ is

(a) $\dfrac{2x+4}{6x-3}$ (b) $\dfrac{6x-4}{2x+3}$ (c) $\dfrac{4x+3}{6-2x}$ (d) Does not exist

$y = \dfrac{6x-3}{2x+4}$

$x \leftrightarrow y$

$x = \dfrac{6y-3}{2y+4}$

$2xy + 4x = 6y - 3$

$y(2x-6) = -4x-3$

$y = \dfrac{-4x-3}{2x-6} = \dfrac{4x+3}{6-2x}$

ODD & EVEN FUNCTIONS :

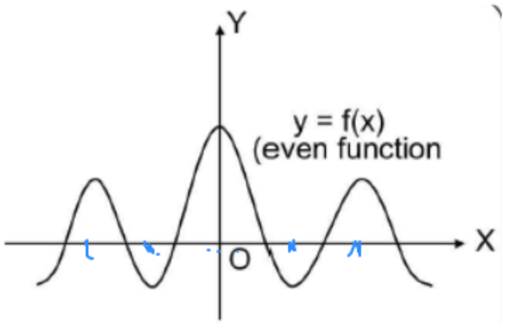
If f (–x) = f (x) for all x in the domain of 'f' then f is said to be an even function.

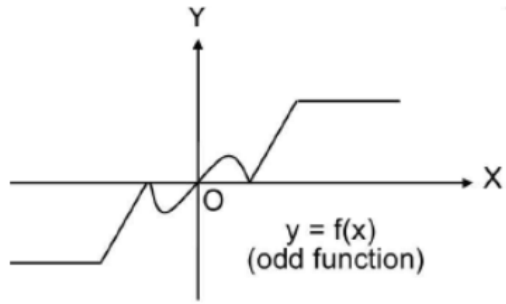
If f (–x) = –f (x) for all x in the domain of 'f' then f is said to be an odd function.

$f(-x) = -f(x)$

$\boxed{f(-x) = f(x)}$

e^x





$$y = \begin{cases} x^2, & x > 0 \\ -x^2, & x \le 0 \end{cases}$$

Even fn are symmetric about y-axis

Odd fn are sym about origin.

e^x , x^3 , $x^3 + 1$

If $f(x) = \dfrac{x+2}{x-3}$; then f (x) is

(a) even function

(b) odd function

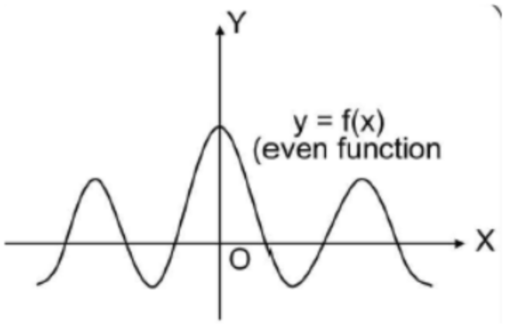
(c) neither even function nor odd function

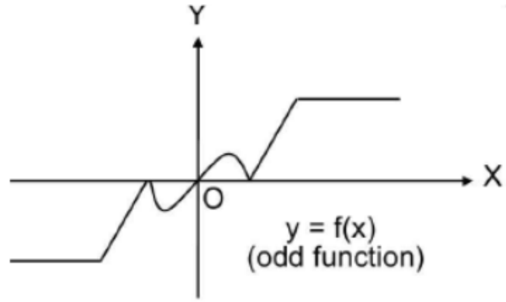
(d) periodic function

ODD & EVEN FUNCTIONS :

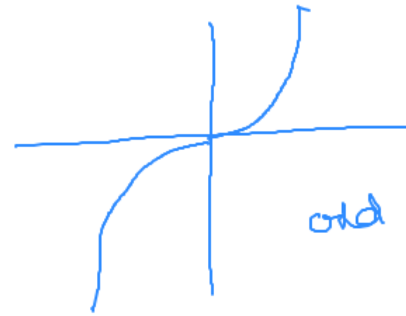
If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.





$$y = \begin{cases} x^2, & x > 0 \\ -x^2, & x \le 0 \end{cases}$$



odd fn

Range $= (-\infty, \infty)$

$$f(-x) = \frac{-x+2}{-x-3} \neq f(x)$$
$$\neq -f(x)$$

If $f(x) = \dfrac{x+2}{x-3}$; then f (x) is

(a) even function

(b) odd function

(c) neither even function nor odd function

(d) periodic function

ODD & EVEN FUNCTIONS :

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.

$f(x) = f(-x)$

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.

one – one

many – one

(a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd .

(b) A function may neither be odd nor even .

(c) Inverse of an even function ~~is not defined~~ .

(d) Every even function is symmetric about the y–axis & every odd function is symmetric about the origin.

(e) Every function can be expressed as the sum of an even & an odd function.

e.g. $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$

$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{odd.}}$

(f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$. $f(x) = 0$

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd .

$f(x) \to$ odd $g(x) \to$ odd $f(x) \times g(x)$

ODD & EVEN FUNCTIONS :

If $f(-x) = f(x)$ for all x in the domain of 'f' then f is said to be an even function.

If $f(-x) = -f(x)$ for all x in the domain of 'f' then f is said to be an odd function.

$f(x)$ $\downarrow$ odd $g(x)$ $\downarrow$ odd

$f(x) \times g(x) = h(x)$

$h(-x) = f(-x) \times g(-x)$

$= -f(x) \times -g(x)$

$= h(x)$

even

(a) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(b) A function may neither be odd nor even.

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$$\text{e.g. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line & is even and odd at the same time is $f(x) = 0$.

(g) If f and g both are even or both are odd then the function f.g will be even but if any one of them is odd then f.g will be odd.

PERIODIC FUNCTION :

A function f(x) is called periodic if there exists a positive number T (T > 0) called the period of the function such that $f(x+T) = f(x)$, for all values of x within the domain of x.

e.g. The function sin x & cos x both are periodic over 2π & tan x is periodic over π.

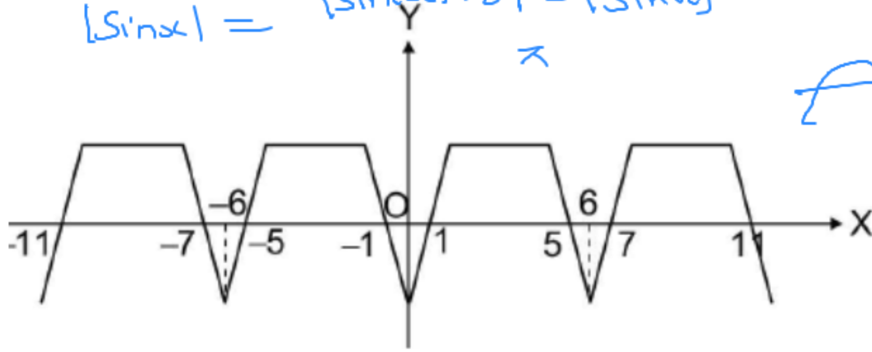
$|\sin x| = |\sin(x+\pi)| = |\sin x|$

π

$f(x+T) = f(x)$

$\sin(x+2\pi) = \sin x$

$\tan(x+\pi) = \tan x$



2π

(a) Inverse of a periodic function does not exist.

(b) Every constant function is always periodic, with no fundamental period.

(c) If f(x) has a period T & g (x) also has a period T then it does not mean that f(x) + g (x) must have a period T . e.g. $f(x) = |\sin x| + |\cos x|$.

$\pi/2$

(d) if f(x) has a period T then f(ax + b) has a period T/a (a > 0).

$f(x+\frac{\pi}{2}) = |\sin(x+\frac{\pi}{2})| + |\cos(x+\frac{\pi}{2})|$

$|\cos x| + |\sin x| = f(x)$

$\sin 2x$ $\quad \frac{2\pi}{2} = \pi$. $\quad \tan(3x+5)$ $\quad T = \pi/3$

Problem

If $f(x) = \cos x + \{x\}$ *where* $\{.\}$ **is fractional part function then the period of** $f(x)$ **is**

a) 2π b) 1   c) $\frac{\pi}{2}$ d) Does not exist

2π 1

L.C.M (1, 2π)

Rational No

Irrational no

L.C.M of Ration & IRR is not defined

$h(x) = f(x) + g(x)$

T_1 T_2

6 $\begin{cases} 2, 4, 6, 8, 10, 12 \\ 3, 6, 9, 12, 15, 18 \end{cases}$

period of $h(x)$ = L.C.M of T_1 & T_2